LECTURE NOTE-3

ON

TOPIC: SEQUENCES AND THEIR CONVERGENCES

SUB-TOPIC: SUBSEQUENCES AND THE BOLZANO WEIRSTRASS THEOREM

WITH SOME APPLICATIONS

FOR

B.Sc. 2^{nd} and 4^{th} SEMESTER (Honours/ Major)

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Introduction

Roughly, a subsequence of a sequence is another sequence obtained by selecting some terms from the given sequence following some specific rule or selection procedure. There is a definite purpose in constructing such subsequences from a given sequence. Mainly, subsequences are often found useful for the purpose of establishing the convergence or divergence of the given sequence. In this discourse, we will state and prove a very important theorem known as the Bolzano-Weirstrass Theorem, which is basically an existence theorem and is also very useful to establish several significant results in analysis.

Subsequences: *Definition*

Let $X = \langle x_n \rangle$ be a sequence of real numbers and $n_1 < n_2 < n_3 < n_4 < .... < n_k < ....$ be a strictly increasing sequence of natural numbers. Then the sequence $X' = \langle x_{n_k} \rangle = \langle x_{n_1}, x_{n_2}, x_{n_3}, ... \rangle$ is called a subsequence of *X*.

For example, consider the sequence $X = \left\langle x_n = \frac{1}{n} : n \in \mathrm{N} \right\rangle$. Then the following are some sequences which are subsequences of the sequence $X = \left\langle x_n = \frac{1}{n} \right\rangle$.

$$\left\langle 1,\ \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8} .... \right\rangle,\quad \left\langle 1,\ \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9} .... \right\rangle,\quad \left\langle 1,\ \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4} .... \right\rangle$$

$$\left\langle 1,\ \frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12} .... \right\rangle,\quad \left\langle 1,\ \frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20} .... \right\rangle,\quad \left\langle 1,\ \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{3^4} .... \right\rangle$$

Theorem 1: If a sequence $X = \langle x_n \rangle$ converges to a real number *x*, then any subsequence of it also converges to *x*.

Proof: Let the real sequence $X = \langle x_n \rangle$ converges to the real number *x* and $X' = \langle x_{n_1}, x_{n_2}, x_{n_3}, ... \rangle$ be a subsequence of it. We prove that the subsequence $X' = \langle x_{n_1}, x_{n_2}, x_{n_3}, ... \rangle$ also converges to *x*.

Let $\varepsilon > 0$ be arbitrary. Then,

$$x_n \to x,\ \varepsilon > 0 \Rightarrow \exists n_0 \in \mathrm{N}\ s.t. \left| x_n - x \right| < \varepsilon,\ \forall\, n \geq n_0$$

Now, $X' = \langle x_{n_1}, x_{n_2}, x_{n_3}, ... \rangle$ is a subsequence $\Rightarrow n_1 < n_2 < n_3 < n_4 < .... < n_k < ....$

$$\Rightarrow \exists n_p \in \mathrm{N}\ s.t.\ \ n_p \geq n_0$$

Then, clearly, $\left|x_{n_k} - x\right| < \varepsilon, \forall n_k \geq n_p \geq n_0$ and hence, $x_{n_k} \to x$.

Application of the above Theorem:

Example 1: If $0 < a < 1$, prove that $\lim (a^n) = 0$.

Proof: Let $X = \langle x_n = a^n : n \in \mathrm{N} \rangle$. Then, we have,

$$0 < a < 1 \Rightarrow 0 < a^{n+1} < a^n, \ \forall \ n \in \mathrm{N} \Rightarrow x_{n+1} < x_n, \ \forall \ n \in \mathrm{N}.$$

So, it is clear that $X = \langle x_n = a^n : n \in \mathrm{N} \rangle$ is a decreasing sequence of real numbers.

Further, $0 < a < 1 \Rightarrow 0 < a^n < 1, \ \forall \ n \in \mathrm{N} \Rightarrow 0 < x_n < 1, \ \forall \ n \in \mathrm{N} \Rightarrow \langle x_n \rangle \ \textit{is bounded}.$

So, by monotone convergence theorem $X = \langle x_n = a^n : n \in \mathrm{N} \rangle$ is convergent. Let $\lim(x_n) = x$.

Again, we have, $x_{2n} = a^{2n} = (a^n)^2 = (x_n)^2, \ \forall \ n \in \mathrm{N}$ and the sequence $X' = \langle x_{2n} \rangle$ is clearly a subsequence of $X = \langle x_n = a^n : n \in \mathrm{N} \rangle$. So, by the above theorem,

$$\begin{aligned} & x = \lim(x_n) = \lim(x_{2n}) = \lim(a^{2n}) = \lim(a^n .\ a^n) = \lim(a^n).\lim(a^n) = x.x = x^2 \\ & \Rightarrow x = x^2 \\ & \Rightarrow x = 0 \ or \ 1 \end{aligned}.$$

Since, the sequence is decreasing and bounded above by 1, so, we have, $x = \lim (a^n) = 0$.

Example 1: If $c > 1$, prove that $\lim (c^{1/n}) = 1$.

Proof: Let $X = \langle x_n = c^{1/n} : n \in \mathrm{N} \rangle$. Then, we have,

$$c > 1 \Rightarrow c^{\frac{1}{n}} > c^{\frac{1}{n+1}}, \ \forall \ n \in \mathrm{N} \Rightarrow x_{n+1} < x_n, \ \forall \ n \in \mathrm{N}.$$

So, it is clear that $X = \left\langle x_n = c^{\frac{1}{n}} : n \in \mathrm{N} \right\rangle$ is a decreasing sequence of real numbers. Further, $c > 1 \Rightarrow c^{\frac{1}{n}} > 1, \ \forall \ n \in \mathrm{N} \Rightarrow x_n > 1, \ \forall \ n \in \mathrm{N} \Rightarrow \langle x_n \rangle \ \textit{is bounded below}.$

So, by monotone convergence theorem $X = \left\langle x_n = c^{\frac{1}{n}} : n \in \mathrm{N} \right\rangle$ is convergent. Let $\lim(x_n) = x$.

Again, we have, $x_{2n} = c^{\frac{1}{2n}} = \left(c^{\frac{1}{n}} \right)^{\frac{1}{2}} = (x_n)^{\frac{1}{2}}, \ \forall \ n \in \mathrm{N}$ and the sequence $X' = \langle x_{2n} \rangle$ is clearly a subsequence of $X = \left\langle x_n = c^{\frac{1}{n}} : n \in \mathrm{N} \right\rangle$. So, by the above theorem,

$$
\begin{aligned}
& x = \lim(x_n) = \lim(x_{2n}) = \lim(x_n)^{\frac{1}{2}} = [\lim(x_n)]^{\frac{1}{2}} = x^{\frac{1}{2}} \\
& \Rightarrow x = x^2 \\
& \Rightarrow x = 0 \ or \ 1
\end{aligned}
$$

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Since, the sequence is decreasing and bounded below by 1, so, we have, $x = \lim\,(c^{1/n}) = 1$.

Negation of the Definition of Convergence:

The following theorem gives the negation of the definition of convergence and it leads to a convenient method to establish the divergence of a sequence.

Theorem: For a sequence $X = \langle x_n \rangle$ of real numbers, the following statements are equivalent.

i) The sequence $X = \langle x_n \rangle$ does not converge to $x \in R$.
ii) There exists an $\varepsilon > 0$ s. t. for any $k \in \mathrm{N}$, $\exists \ n_k \in \mathrm{N}$ s. t. $n_k > k$ and $|x_{n_k} - x| \geq \varepsilon$
iii) There exists an $\varepsilon > 0$ and a subsequence $X' = \langle x_{n_k} \rangle$ of the sequence $X = \langle x_n \rangle$ such that $|x_{n_k} - x| \geq \varepsilon$ for all $k \in \mathrm{N}$.

Divergence criteria

If a sequence $X = \langle x_n \rangle$ of real numbers has either of the following properties, then X is divergent.

i) X has two convergent subsequences whose limits are not equal.
ii) X is unbounded.

Applications:

In the following examples, we will see nice applications of the above divergence criteria.

Examples:

A) The sequence $X = \langle (-1)^n \rangle = \langle -1, 1, -1, 1, -1, \ldots\ldots\ldots \rangle$ is divergent.

Proof: Here, the subsequences $X' = \langle -1, -1, -1, -1, \ldots\ldots \rangle$ and $X'' = \langle 1, 1, 1, \ldots\ldots \rangle$ of the sequence $X = \langle (-1)^n \rangle = \langle -1, 1, -1, \ldots\ldots \rangle$ are convergent and converge to the limits -1 and 1 respectively. Thus two convergent subsequences of the given sequence converge to two different limits. So, by divergence criteria, the sequence X must be divergent.

B) The sequence $X = \left\langle 1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots\ldots \right\rangle$ is divergent.

Proof: Here the sequence $X = \left\langle 1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots\ldots \right\rangle$ is such that

$$x_n = n \ \ for\ n\ odd \ \ and \ \ x_n = \frac{1}{n} \ \ for\ n\ even$$

For odd n, the values of the terms will increase abundantly. So, the sequence is unbounded and hence by divergence criteria, it is divergent.

C) The sequence $X = \langle \sin(n) : n \in \mathrm{N} \rangle$ is divergent. (H.W.)

The existence of monotone sequences:

Though every real sequence is not a monotone sequence, we can show that every such sequence has a monotone subsequence. The following theorem establishes this fact clearly.

The Monotone Subsequence Theorem: Every sequence of real numbers has a monotone subsequence.

Proof: Let $X = \langle x_n : n \in \mathrm{N} \rangle$ be a real sequence. We now construct a monotone subsequence of this sequence.

Let us first define a term "peak" which will help for our purpose.

If $x_m \geq x_n$, $\forall\, n \geq m$, then x_m is called a peak. That is, if for some $m \in \mathrm{N}$, all the terms of the sequence after m-terms never exceeds x_m, then it is a peak. Here, we note that, by definition, every term of a decreasing sequence is always a peak and no term of an increasing sequence is a peak. Then, depending on the number of peaks, there may arise two cases: Case I: The sequence has infinitely many peaks and Case II: the sequence has finite number of peaks.

Case I: Let $X = \langle x_n : n \in \mathrm{N} \rangle$ has infinitely many peaks and let the peaks, after listing them by increasing subscripts, be x_{m_1}, x_{m_2}, x_{m_3}, x_{m_4}, ... Since, each term of this list is a peak, so,

$$x_{m_1} > x_{m_2} > x_{m_3} > x_{m_4} > x_{m_5} > x_{m_6} > x_{m_7} > \ldots\ldots\ldots\ldots\ldots\ldots\ldots .$$

Therefore, the subsequence $\langle x_{m_1}, x_{m_2}, x_{m_3}, x_{m_4}, \ldots\ldots\ldots \rangle = \langle x_{m_n} : n \in \mathrm{N} \rangle$ of peaks obtained in this way is a decreasing subsequence of *X*.

Case II: Let $X = \langle x_n : n \in \mathrm{N} \rangle$ has finite number of peaks including no peak at all.

Let, in increasing subscripts, the peaks are listed as $x_{m_1}, x_{m_2}, x_{m_3}, \ldots\ldots, x_{m_k}$. Further, let, $s_1 = m_r + 1$ be the first index beyond the last peak. Then, x_{s_1} is not a peak and hence there exists $s_2 > s_1$ such that $x_{s_1} < x_{s_2}$. Since, x_{s_2} is not a peak, so there exists $s_3 > s_2$ such that $x_{s_2} < x_{s_3}$. Continuing in this way, we obtain an increasing subsequence $\langle x_{s_n} : n \in \mathrm{N} \rangle$ of *X*.

Note: The above theorem can be applied to prove Bolzano-Weirstrass Theorem.

The Bolzano-Weirstrass Theorem: *A bounded sequence of real numbers has a convergent subsequence.*

First Proof: Using Monotone Subsequence Theorem:

Let $X = \langle x_n : n \in \mathrm{N} \rangle$ be a bounded real sequence. By the Monotone Subsequence Theorem, the sequence *X* has a monotonic subsequence $X' = \langle x_{n_k} \rangle$. Since *X* is a bounded sequence, so, every subsequence of it is clearly bounded and hence $X' = \langle x_{n_k} \rangle$ is bounded. Thus $X' = \langle x_{n_k} \rangle$ is a bounded monotonic sequence. Therefore, by the Monotone Convergence Theorem, $\langle x_{n_k} \rangle$ is convergent. Hence, the theorem follows.

Second Proof: Using nested interval Theorem:

Let $X = \langle x_n : n \in \mathrm{N} \rangle$ be a bounded real sequence. Then, there an interval $I_1 = [a, b]$ such that $X = \langle x_n : n \in \mathrm{N} \rangle \subset I = [a, b]$. Take $n_1 = 1$.

Bisect the interval $I_1 = [a, b]$ into two subintervals I_{11} and I_{12}. Also, let us divide the set of indices $\{n \in \mathrm{N} : n > 1\}$ into two parts:

$$A_1 = \{n \in \mathrm{N} : n > n_1,\ x_n \in I_{11}\},\ \ B_1 = \{n \in \mathrm{N} : n > n_1,\ x_n \in I_{12}\}$$

Then, at least one of A_1 & B_1 is infinite. W.L.O.G. (without loss of generality), let, B_1 be infinite. Then, label $I_{12} = I_2$ and let n_2 be the smallest natural number in B_1.

Again, bisect the interval I_2 into two subintervals I_{21} and I_{22}. Also, let us divide the set of natural numbers $\{n \in \mathrm{N} : n > n_2\}$ into two parts:

$$A_2 = \{n \in \mathrm{N} : n > n_2,\ x_n \in I_{21}\},\ \ B_2 = \{n \in \mathrm{N} : n > n_2,\ x_n \in I_{22}\}$$

Then, at least one of A_2 & B_2 is infinite. W.L.O.G. (without loss of generality), let, A_2 be infinite. Then, label $I_{21} = I_3$ and let n_3 be the smallest natural number in A_2.

Continuing in this process, we obtain a nested sequence $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots\ldots\ldots\ldots$ of intervals and a subsequence $\langle x_{n_k} : n_k \in \mathrm{N} \rangle$ of X such that $x_{n_k} \in I_k$ for all $k \in \mathrm{N}$. Here, we note that $|I_k| = (b-a)/2^{k-1}$. Then, by nested interval property, there exists a unique common point $x \in I_k,\ \forall\ k \in \mathrm{N}$.

Now, $x_{n_k},\ x \in I_k,\ \forall k \in \mathrm{N} \Rightarrow |x_{n_k} - x| \le (b-a)/2^{k-1} \to 0\ as\ k \to \infty$

So, it follows that $x_{n_k} \to x\ \ as\ k \to \infty$. i.e., the subsequence $\langle x_{n_k} : n_k \in \mathrm{N} \rangle$ is convergent. Hence the theorem follows.

Theorem: *If a bounded sequence* $X = \langle x_n : n \in \mathrm{N} \rangle$ *of real numbers be such that every convergent subsequence of it converges to x, then the sequence X itself converges to x.*

Proof: Let $M > 0$ be a bound for the sequence X such that $|x_n| \le M,\ \forall n \in \mathrm{N}$.

Now, if $X = \langle x_n : n \in \mathrm{N} \rangle$ does not converge to *x*, then, there exists $\varepsilon > 0$ and subsequence $X' = \langle x_{n_k} \rangle$ of *X* such that $|x_{n_k} - x| \ge \varepsilon,\ \ \forall k \in \mathrm{N} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$.

Since, $X' = \langle x_{n_k} \rangle$ is a subsequence of *X*, the number $M > 0$ is also a bound for $X' = \langle x_{n_k} \rangle$. Hence, by Bolzano-Weirstrass Theorem, $X' = \langle x_{n_k} \rangle$ has convergent subsequence X'', say. Then, it is also a convergent subsequence of *X* and so, by our hypothesis, X'' converges to *x*.

Thus its terms ultimately belong to the ε - neighbourhood of *x* which contradicts (1).

So, the theorem follows.