LECTURE NOTE-2

ON

TOPIC: SEQUENCES AND THEIR CONVERGENCES

SUB-TOPIC: MONOTONIC SEQUENCES AND MONOTONE CONVERGENCE THEOREM

WITH SOME APPLICATIONS

FOR

B.Sc. 2^{nd} and 4^{th} SEMESTER (Honours/ Major)

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Introduction.

Monotonicity and boundedness are two very strong characteristics in the context of testing convergence and divergence of real sequences. We have a very important theorem, namely, the monotone convergence theorem, important in the sense of its usefulness and application, which not only ensures us that a bounded monotonic sequence of real numbers is always convergent, but also unambiguously points out the fact that such a sequence always converges to its lub (least upper bound or supremum)/glb (greatest lower bound or infimum) depending on the monotonicity nature of the particular sequence. So, it is considered as a strong tool of analysis for handling the problem of testing convergence/divergence of monotonic sequences.

In this discourse, we will mainly discuss about monotonic sequences and the very useful monotonic convergence theorem in detail. We will also deal with some particular problems where monotone convergence theorem can be applied very efficiently.

Definition.1. A real sequence $X = \langle x_n \rangle$ is said to be

i) An increasing sequence if $x_1 < x_2 < x_3 < .... < x_n < ....$ i.e., if $x_n < x_{n+1}, \forall n \in \mathrm{N}$.

ii) A decreasing sequence if $x_1 > x_2 > x_3 > .... > x_n > ....$ i.e., if $x_n > x_{n+1}, \forall n \in \mathrm{N}$.

iii) A monotonic increasing sequence if $x_1 \le x_2 \le x_3 \le x_4 \le .... \le x_n \le ....$ i.e., if $x_n \le x_{n+1}, \forall n \in \mathrm{N}$.

iv) A monotonic decreasing sequence if $x_1 \ge x_2 \ge x_3 \ge x_4 \ge .... \ge x_n \ge ....$ i.e., if $x_n \ge x_{n+1}, \forall n \in \mathrm{N}$.

v) A monotonic sequence if it is either monotonic increasing or monotonic decreasing.

Note:

a) From the above definitions it is clear that an increasing sequence is always monotonic increasing. But a monotonic increasing sequence is not necessarily an increasing sequence. Similarly, a decreasing sequence is always monotonic decreasing. But a monotonic decreasing sequence is not necessarily decreasing.

b) If $X = \langle x_n \rangle$ is increasing/monotonic increasing, then, clearly the sequence $-X = \langle -x_n \rangle$ is decreasing/monotonic decreasing. Similarly, If the sequence $X = \langle x_n \rangle$ is monotonic decreasing/ decreasing, then, $-X = \langle -x_n \rangle$ is monotonic decreasing/ increasing.

Example.1: The following sequences are Monotonic increasing sequences:

i) The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, ...$

ii) The constant sequences like $k, k, k, k, k, k, k, ...,\ where\ \ k \in R$

iii) $1, \ 3, 5, 7, \ 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, \ ....$

iv) $1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, 10, 10, 11, 11, ...$

v) $1, \ 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, \ ....$

Example.2: The following sequences are Monotonic decreasing sequences:

i) $1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9, 1/10, 1/11, 1/12, \ldots\ldots\ldots\ldots$

ii) $-1, -2, -3, -4, -5, -6, -7, -8, -9, -10, -11, -12, -13, \ldots\ldots\ldots$

iii) $a, a^2, a^2, a^3, a^3, a^3, a^4, a^4, a^4, a^4, a^5, a^5, a^5, a^5, a^5, \ldots\ldots.$ *where* $0 < a < 1$

iv) $1, 1, 1/2, 1/2, 1/3, 1/3, 1/4, 1/4, 1/5, 1/5, 1/6, 1/6, 1/7, 1/7, \ldots\ldots\ldots$

v) $1, \frac{1}{2^2}, \frac{1}{3^3}, \frac{1}{4^4}, \frac{1}{5^5}, \frac{1}{6^6}, \frac{1}{7^7}, \frac{1}{8^8}, \frac{1}{9^9}, \ldots\ldots\ldots.$

Example.3: The following sequences are neither increasing nor decreasing:

i) $1, -2, 3, -4, 5, -6, 7, -8, 9, -10, 11, -12, 13, \ldots\ldots\ldots$

ii) $1, 0, -1, 1, 0, -1, 1, 0, -1, 1, 0, -1, 1, 0, -1, \ldots\ldots\ldots$

iii) $1, -1, 1, -1, 1, -1, 1, -1, 1, -1, . 1, -1, 1, -1, \ldots\ldots\ldots$

Example.4: The following sequences are not monotonic, but they are ultimately monotonic (i.e., monotonic after a certain number of terms):

i) 5, 4, 7, 2, 13, 6, 10, 14, 18, 22, 26, 30, 34, 38,………………

ii) 4, 3, 2, 11, 1, 1/2, 1/3, 1/4, 1/5, 1/6, …………………………

iii) 100, 78, 256, 34, 99, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5,………………

<u>The monotone Convergence Theorem:</u> *A real monotonic sequence is convergent if and only if it is bounded. Further, if* $X = \langle x_n \rangle$ *is bounded and monotonic*

i) *Increasing, then,* $\lim\langle x_n \rangle = \sup\{x_n : n \in \mathrm{N}\}$

ii) *Decreasing, then,* $\lim\langle x_n \rangle = \inf\{x_n : n \in \mathrm{N}\}$

Proof: We have already proved that a convergent sequence is always bounded. So, one part of the proof is already done.

Conversely, let, $X = \langle x_n \rangle$ be bounded. Then there may arise two cases: (i) $\langle x_n \rangle$ is monotonically increasing or (ii) $\langle x_n \rangle$ is monotonically decreasing.

Case (i): Let $\langle x_n \rangle$ be monotonically increasing.

Now, $X = \langle x_n \rangle$ is bounded $\Rightarrow \exists M \in R \ s.t. \ x_n \le M, \forall n \in \mathrm{N}$.

Then, by the completeness Property, the supremum $x^* = \sup\{x_n : n \in \mathrm{N}\}$ must belong to R.

We now show that $\lim\langle x_n \rangle = x^* = \sup\{x_n : n \in \mathrm{N}\}$.

Let $\varepsilon > 0$ be arbitrary. Then,

$$\varepsilon > 0,\ x^* = \sup\{x_n : n \in \mathrm{N}\} \Rightarrow x^* - \varepsilon \textit{ is not an upper bound of } X = \langle x_n \rangle$$
$$\Rightarrow \exists \textit{at least one } n_0 \in \mathrm{N} \textit{ s. t. } x^* - \varepsilon < x_{n_0}$$

Since, $X = \langle x_n \rangle$ is monotonic increasing, so, $x^* - \varepsilon < x_n,\ \forall n \ge n_0$.

Further, we have,

$$x^* - \varepsilon < x_{n_0} \le x_n \le x^* < x^* + \varepsilon, \quad \forall n \ge n_0$$
$$\Rightarrow x^* - \varepsilon < x_n < x^* + \varepsilon, \quad \forall n \ge n_0$$
$$\Rightarrow \left| x_n - x^* \right| < \varepsilon, \quad \forall n \ge n_0$$
$$\Rightarrow \lim (x_n) = x^* = \sup\{x_n : n \in \mathrm{N}\}$$

Case (ii): Let $\langle x_n \rangle$ be monotonically decreasing so that $x_n \ge x_{n+1},\ \forall n \in \mathrm{N}$.

Now, $X = \langle x_n \rangle$ is bounded $\Rightarrow \exists b \in R \ s.t.\ x_n \ge b,\ \forall n \in \mathrm{N}$.

Then, by the completeness Property, the supremum $x^* = \inf\{x_n : n \in \mathrm{N}\}$ must belong to R.

We now show that $\lim\langle x_n \rangle = x^* = \inf\{x_n : n \in \mathrm{N}\}$. Let $\varepsilon > 0$ be arbitrary. Then, we have,

$$\varepsilon > 0,\ x^* = \inf\{x_n : n \in \mathrm{N}\} \Rightarrow x^* + \varepsilon \textit{ is not a lower bound of } X = \langle x_n \rangle$$
$$\Rightarrow \exists \textit{at least one } n_0 \in \mathrm{N} \textit{ s. t. } x^* + \varepsilon > x_{n_0}$$

Since, $X = \langle x_n \rangle$ is monotonic decreasing, so, $x^* + \varepsilon > x_{n_0} \ge x_n,\ \forall n \ge n_0$.

Further, we have,

$$x^* - \varepsilon \le x_n - \varepsilon < x_n \le x_{n_0} \le x^* + \varepsilon, \quad \forall n \ge n_0$$
$$\Rightarrow x^* - \varepsilon < x_n < x^* + \varepsilon, \quad \forall n \ge n_0$$
$$\Rightarrow \left| x_n - x^* \right| < \varepsilon, \quad \forall n \ge n_0$$
$$\Rightarrow \lim (x_n) = x^* = \inf\{x_n : n \in \mathrm{N}\}$$

REMARKS: The monotone convergence theorem confirms that there is a limit for every bounded monotone sequence. It also gives us a way of calculating the limit of the sequence provided we can evaluate the supremum (in case of monotonic increasing sequence)/ infimum (in case of monotonic increasing sequence). Sometimes it is very much difficult to evaluate the limit. But if we know that the limit exists, then we can somehow evaluate it.

Example. 1.: $\lim\left(\frac{1}{\sqrt{n}}\right)=0$

Here, $n\geq 1,\ \forall n\in \mathrm{N}\Rightarrow \sqrt{n}\geq 1$ *and hence* $\frac{1}{\sqrt{n}}\leq 1$. So, $\left\langle \frac{1}{\sqrt{n}}\right\rangle$ is bounded.

Further,

$$n+1>n,\quad \forall n\in \mathrm{N}\Rightarrow \sqrt{n+1}\geq \sqrt{n}$$
$$\Rightarrow \frac{1}{\sqrt{n}}\geq \frac{1}{\sqrt{n+1}}$$

So, it follows that the sequence $X=\left\langle \frac{1}{\sqrt{n}}\right\rangle$ is a monotonic decreasing sequence. Hence, it is a convergent sequence.

Let $\lim X=\lim\left(\frac{1}{\sqrt{n}}\right)$. Then by limit theorem, we have,

$$\lim X.X=\lim\left(\frac{1}{\sqrt{n}}.\frac{1}{\sqrt{n}}\right)=\lim\left(\frac{1}{n}.\right)=0$$
$$\Rightarrow \lim X.\lim X=[\lim X]^2=0$$
$$\Rightarrow \lim X=\lim\left\langle \frac{1}{\sqrt{n}}\right\rangle=0$$

<u>Example 2</u>: Prove that $X=\langle x_n:\ x_n=1+1/2+1/3+1/4+.....+1/n,\ n\in \mathrm{N}\rangle$ is divergent.

Here, $x_{n+1}=1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}+\frac{1}{n+1}=\left[1+\frac{1}{2}+\frac{1}{3}+........+\frac{1}{n}\right]+\frac{1}{n+1}=x_n+\frac{1}{n+1}>x_n$

So, it is a monotonic increasing sequence. But we have,

$$x_{2^n}=1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{2^n}=1+\frac{1}{2}+\left[\frac{1}{3}+\frac{1}{4}\right]+..........+\left[\frac{1}{2^{n-1}+1}+\frac{1}{2^{n-1}+2}+....+\frac{1}{2^n}\right]$$
$$>1+\frac{1}{2}+\frac{1}{2}+.......+\frac{1}{2}=1+\frac{n}{2}$$

Since, $X=\langle x_n\rangle$ is unbounded, so it follows that it is divergent.

REMARKS: Inductively defined sequences are treated differently as the limits of such sequences cannot be evaluated so much easily. If we know that such sequences are convergent, then, in most of the cases the limits are calculated by using the inductive relations.

Example 1: Suppose that the sequence $X = \left\langle x_n : x_1 = 2,\ x_{n+1} = 2 + \frac{1}{x_n},\ n \in \mathrm{N} \right\rangle$ be convergent. Find the limit.

Solution: Let $\lim X = \lim\langle x_n \rangle = x$

Then, since, the limit of a convergent sequence is same as the limit of its m-tail (i.e., the sequence from the (m+1)-th term), so,

$$x = \lim\langle x_n \rangle = \lim\langle x_{n+1} \rangle = \lim\left\langle 2 + \frac{1}{x_n} \right\rangle = \left\langle 2 + \frac{1}{\lim(x_n)} \right\rangle = 2 + \frac{1}{x}$$

$$\Rightarrow x = 2 + \frac{1}{x} \Rightarrow x^2 - 2x - 1 = 0 \Rightarrow x = 1 + \sqrt{2} \quad [\because x > 0] \quad (why???)$$

So, the limit of the sequence is $1 + \sqrt{2}$.

Example 2: Prove that the sequence $X = \left\langle x_n : x_1 = 1,\ x_{n+1} = \frac{1}{4}(2x_n + 3),\ n \in \mathrm{N} \right\rangle$ is convergent and its limit is 3/2.

Proof: We first prove that the given sequence is bounded above and monotonic increasing.

(i) $X = \langle x_n \rangle$ is bounded above:

By direct calculation, we have,

$$x_2 = 5/2 < 2 \text{ and so, } x_1 < x_2 < 2$$

Let $x_k < 2$ for some $k \in \mathrm{N}$. Then, $x_{k+1} = \frac{1}{4}(2x_k + 3) < \frac{1}{4}(2.2 + 3) = \frac{7}{4} < 2$

So, by induction, $x_n < 2,\ \forall n \in \mathrm{N}$. Hence, the sequence is bounded above.

(ii) $X = \langle x_n \rangle$ is monotonic increasing:

We have already found that $x_1 < x_2 < 2$. Let $x_k < x_{k+1}$ for some $k \in \mathrm{N}$ so that $2x_k + 3 < 2x_{k+1} + 3$.

Then, $x_{k+1} = \frac{1}{4}(2x_k + 3) < \frac{1}{4}(2x_{k+1} + 3) = x_{k+2}$. Thus, $x_k < x_{k+1} \Rightarrow x_{k+1} < x_{k+2}$.

So, by induction, we can conclude that the sequence is monotonic increasing.

Therefore, by monotone convergence theorem, it is convergent and converges to its supremum.

Finally, we prove that $\lim X = \lim\langle x_n \rangle = \sup\langle x_n \rangle$

If we assume that $\lim X = \lim\langle x_n \rangle = x$, then, we have,

$$x = \lim\langle x_n \rangle = \lim\langle x_{n+1} \rangle = \lim\left\langle \frac{1}{4}(2x_n + 3) \right\rangle = \left\langle \frac{1}{4}\left(2.\lim(x_n) + 3\right) \right\rangle = \frac{1}{4}\left(2x + 3\right)$$

$$\Rightarrow 4x = 2x + 3 \Rightarrow x = \frac{3}{2}$$

So, the given sequence converges to the limit 3/2.