MATH.M6SEM20NA

CLASSNOTES

ON

NUMERICAL ANALYSIS

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CHAPTER 4

CENTRAL DIFFERENCE INTERPOLATION FORMULAE

4.01. Introduction

We have already discussed Newton-Gregory's interpolation formulae (forward and backward) for entries at equidistant values of the argument and also have solved some numerical problems related to these formulae. These formulae are very much fundamental and applicable to nearly all classes of problems related to interpolations. But these are more applicable to estimate values of the function near the beginning and the end of the set of tabulated values of the argument. The main drawback of these formulae is that they, in general, do not converge more rapidly. We have also studied Newton and Lagrange's divided difference formulae for unequal intervals of the argument which are of more generalized nature (because one can derive the formulae for equal intervals from these). These formulae are more tedious to apply as they involve a great deal of computations. Further, if the arguments are quite apart, then these formulae do not give accurate results. So, we need some other kind of interpolation formulae which are free from all the above mentioned demerits. Central difference interpolation formulae are some such type of formulae.

The central difference formulae, as the name indicates, are more suitable for interpolating values of the function near the middle of a tabulated set of values. The main advantages of these formulae are: (i) they converge more rapidly and (ii) the origin may be shifted to some other convenient point so as to simplify the situations.

4.02 The central difference operators δ, μ and σ:

4.02.1 The first order central difference of $f(x)$ is denoted by $\delta f(x)$ and is defined by

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right) = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) f(x)\text{ , where } E \equiv 1 + \Delta$$

Thus $\delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}} = E^{-\frac{1}{2}}(E-1) = E^{-\frac{1}{2}}\Delta = \Delta E^{-\frac{1}{2}}$

Therefore, $\delta E^{\frac{1}{2}} f(x) = \left[\left(\Delta E^{-\frac{1}{2}}\right) E^{\frac{1}{2}}\right] f(x) = \left[\Delta(E^{-\frac{1}{2}} E^{\frac{1}{2}})\right] f(x) = \Delta f(x)$

And so, $\delta f\left(x+\frac{h}{2}\right) = f(x+h) - f(x) = \Delta f(x)$

Further, $\nabla f(x) = f(x) - f(x-h) = (1-E^{-1}) f(x)$

$$= \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) E^{-\frac{1}{2}} f(x) = \delta E^{-\frac{1}{2}} f(x)$$

Thus $\nabla \equiv \delta E^{-\frac{1}{2}}$. That is, $\delta \equiv \nabla E^{\frac{1}{2}}$

Ultimately, we have found that

$$\delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}} = \nabla E^{\frac{1}{2}} = \Delta E^{-\frac{1}{2}}$$

The nth order central difference as applied to $f(x) = y_x$ is obtained as

$$\delta^n y_x = \nabla^n E^{\frac{n}{2}} y_x = \Delta^n E^{\frac{-n}{2}} y_x$$

i.e, $\delta^n y_x = \nabla^n y_{x+n/2} = \Delta^n y_{x-n/2}$

4.02.2 The average operator μ is defined by the operator equation

$$\mu = \frac{\mathbf{1}}{\mathbf{2}}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) \text{ so that } \mu y_x = \frac{\mathbf{1}}{\mathbf{2}}\left(y_{x+1/2} + y_{x-1/2}\right)$$

Now $\mu^{\mathbf{2}} y_x = \left[\frac{\mathbf{1}}{\mathbf{2}}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)\right]^{\mathbf{2}} y_x = \frac{\mathbf{1}}{\mathbf{4}}\left[\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^{\mathbf{2}} + 4\right] y_x$

$$= \frac{\mathbf{1}}{\mathbf{4}}[\delta^{\mathbf{2}} + 4] y_x = \left[\mathbf{1} + \frac{\mathbf{1}}{\mathbf{4}}\delta^{\mathbf{2}}\right] y_x$$

Therefore, $\mu^{\mathbf{2}} \equiv \mathbf{1} + \frac{\mathbf{1}}{\mathbf{4}}\delta^{\mathbf{2}}$

Again from the relations $\mu y_x = \frac{\mathbf{1}}{\mathbf{2}}\left(y_{x+1/2} + y_{x-1/2}\right)$ and $\delta y_x = y_{x+1/2} - y_{x-1/2}$ we have,

$$2\mu y_x + \delta y_x = 2y_{x+1/2} \text{ i.e., } (2\mu + \delta) y_x = 2E^{\frac{1}{2}} y_x$$

So, we have, $2\mu + \delta \equiv 2E^{\frac{1}{2}} \Longrightarrow E^{\frac{1}{2}} \equiv \mu + \frac{\mathbf{1}}{\mathbf{2}}\delta$.

4.02.3 The inverse of the central difference operators δ is denoted by σ so that $\delta\sigma f(x) = f(x)$

Then, $\sigma f(x) = \sigma f(x-h) + f\left(x - \frac{h}{2}\right)$

$$\Longrightarrow \sigma f(x) = \sigma E^{-1} f(x) + E^{-\frac{1}{2}} f(x)$$

$$\Longrightarrow (\sigma - \sigma E^{-1}) f(x) = E^{-\frac{1}{2}} f(x)$$

$$\Longrightarrow \sigma(1 - E^{-1}) \equiv E^{-\frac{1}{2}}$$

$$\Longrightarrow \sigma \equiv \frac{E^{\frac{1}{2}}}{E-1}$$

Further, $f(x) = \sigma f\left(x + \frac{h}{2}\right) - \sigma f\left(x - \frac{h}{2}\right)$.

4.03 Central Difference table:

The appearance of a central difference table for the operator δ is as follows:

x	y_x	δy_x	$\delta^2 y_x$	$\delta^3 y_x$	$\delta^4 y_x$	$\delta^5 y_x$	$\delta^6 y_x$
-3	y_{-3}						
		$\delta y_{-5/2}$					
-2	y_{-2}		$\delta^2 y_{-2}$				
		$\delta y_{-3/2}$		$\delta^3 y_{-3/2}$			
-1	y_{-1}		$\delta^2 y_{-1}$		$\delta^4 y_{-1}$		
		$\delta y_{-1/2}$		$\delta^3 y_{-1/2}$		$\delta^5 y_{-1/2}$	
0	y_0		$\delta^2 y_0$		$\delta^4 y_0$		$\delta^6 y_0$
		$\delta y_{1/2}$		$\delta^3 y_{1/2}$		$\delta^5 y_{1/2}$	
1	y_1		$\delta^2 y_1$		$\delta^4 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
2	y_2		$\delta^2 y_2$				
		$\delta y_{5/2}$					
3	y_3						

Note: (i) The operators δ and σ do not commute. i.e., $\delta\sigma \neq \sigma\delta$.

(ii) $\delta y_{1/2} = \left(\Delta E^{-\frac{1}{2}}\right) y_{\frac{1}{2}} = \Delta(E^{-\frac{1}{2}} y_{\frac{1}{2}}) = \Delta y_0$

$$\delta^2 y_0 = \delta(\delta y_0) = \delta\left(\Delta E^{-\frac{1}{2}}\right) y_0 = \delta \Delta y_{-1/2} = \left(\Delta E^{-\frac{1}{2}}\right) \Delta y_{-1/2} = \Delta^2 y_{-1}$$

and in a similar way, we have, $\delta^3 y_{1/2} = \Delta^3 y_{-1}$

$\delta^4 y_0 = \Delta^4 y_{-2}$ etc.

4.04.01 Gauss's Forward Interpolation Formula:

Let the function y_u takes the values $y_{-3}, y_{-2}, y_{-1}, y_0, y_1, y_2, y_3$ for equally spaced values and unit intervals of u where $u = \frac{x-x_0}{h}$.

Now, from Newton's advancing difference formula, we have,

$$y_u = P_n(a+hu) = y_0 + {}^uC_1 \Delta y_0 + {}^uC_2 \Delta^2 y_0 + {}^uC_3 \Delta^3 y_0 + {}^uC_4 \Delta^4 y_0 + \ldots\ldots\ldots\ldots\ldots$$

But, we have,

$$\Delta^n y_x - \Delta^n y_{x-1} = \Delta^n (y_x - y_{x-1}) = .\Delta^n (\Delta y_{x-1}.) = \Delta^{n+1} y_{x-1},\ \forall n \in \mathrm{N}$$

$$\text{So, } \Delta^n y_x = \Delta^n y_{x-1} + \Delta^{n+1} y_{x-1},\ \forall n \in \mathrm{N}.$$

Thus

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1},\ \ \Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1},\ \ \Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2},$$
$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2},\ \ \Delta^4 y_{-2} = \Delta^4 y_{-3} + \Delta^5 y_{-3},\ \Delta^5 y_{-2} = \Delta^5 y_{-3} + \Delta^6 y_{-3},\ etc.$$

So,

$$\begin{aligned} y_u &= y_0 + {}^uC_1 \Delta y_0 + {}^uC_2 (\Delta^2 y_{-1} + \Delta^3 y_{-1}) + {}^uC_3 (\Delta^3 y_{-1} + \Delta^4 y_{-1}) + {}^uC_4 (\Delta^4 y_{-1} + \Delta^5 y_{-1}). + \ldots\ldots \\ &= y_0 + {}^uC_1 \Delta y_0 + {}^uC_2 \Delta^2 y_{-1} + ({}^uC_2 + {}^uC_3) \Delta^3 y_{-1} + {}^uC_3 (\Delta^4 y_{-2} + \Delta^5 y_{-2}) + {}^uC_4 (\Delta^4 y_{-2} + \Delta^5 y_{-2} + \Delta^5 y_{-1}). + \ldots. \\ &= y_0 + {}^uC_1 \Delta y_0 + {}^uC_2 \Delta^2 y_{-1} + {}^{u+1}C_3 \Delta^3 y_{-1} + {}^{u+1}C_4 \Delta^4 y_{-2} + \ldots. + {}^{u+k-1}C_{2k-1} \Delta^{2k-1} y_{-k+1} + {}^{u+k-1}C_{2k} \Delta^{2k} y_{-k} + \ldots\ldots. \end{aligned}$$

That is,

$$y_u = y_0 + {}^uC_1 \Delta y_0 + {}^uC_2 \Delta^2 y_{-1} + {}^{u+1}C_3 \Delta^3 y_{-1} + {}^{u+1}C_4 \Delta^4 y_{-2} + \ldots. + {}^{u+k-1}C_{2k-1} \Delta^{2k-1} y_{-k+1} + {}^{u+k-1}C_{2k} \Delta^{2k} y_{-k} + \ldots$$

This is Gauss's forward interpolation formula for equal intervals. This formula employs the odd differences just below the central line from y_0 and the even differences on the central line.

Note: In Gauss's interpolation formula, we observe that

(i) The indices of the operator Δ in the terms are in ascending order of units starting from 0.
(ii) The suffixes of the entries, i.e. of y, appear as 0, 0,-1,-1,-2,-2,-3,-3,……. etc.
(iii) $T_{2r} = {}^{u+r-1}C_{2r-1} \Delta^{2r-1} y_{-r+1},\ T_{2r+1} = {}^{u+r-1}C_{2r} \Delta^{2r} y_{-r}\ \ for\, r \ge 1$

4.04.02 Gauss's backward interpolation Formula:

This formula has been derived from Gauss's forward interpolation formula as follows:

Gauss's forward interpolation formula is-

$$y_u = y_0 + {}^uC_1\Delta y_0 + {}^uC_2\Delta^2 y_{-1} + {}^{u+1}C_3\Delta^3 y_{-1} + {}^{u+1}C_4\Delta^4 y_{-2} + ... + {}^{u+k-1}C_{2k-1}\Delta^{2k-1} y_{-k+1} + {}^{u+k-1}C_{2k}\Delta^{2k} y_{-k} + ...$$

Now putting

$\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1},\ \ \Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2},\ \ \Delta^5 y_{-2} = \Delta^5 y_{-3} + \Delta^6 y_{-3}$ and so on in the above Gauss's forward formula and grouping the terms, we get,

$$\begin{aligned} y_u &= y_0 + {}^uC_1(\Delta y_{-1} + \Delta^2 y_{-1}) + {}^uC_2\Delta^2 y_{-1} + {}^{u+1}C_3(\Delta^3 y_{-2} + \Delta^4 y_{-2}) + {}^{u+1}C_4\Delta^4 y_{-2} + ... \\ &\quad + {}^{u+k-1}C_{2k-1}(\Delta^{2k-1} y_{-k+1} + \Delta^{2k} y_{-k}) + {}^{u+k-1}C_{2k}\Delta^{2k} y_{-k} + ... \\ &= y_0 + {}^uC_1\Delta y_{-1} + {}^{u+1}C_2\Delta^2 y_{-1} + {}^{u+1}C_3\Delta^3 y_{-2} + {}^{u+2}C_4\Delta^4 y_{-2} + ... + {}^{u+k-1}C_{2k-1}\Delta^{2k-1} y_{-k} + {}^{u+k}C_{2k}\Delta^{2k} y_{-k} + ... \\ &= y_0 + {}^uC_1\delta y_{-1/2} + {}^{u+1}C_2\delta^2 y_0 + {}^{u+1}C_3\delta^3 y_{-1/2} + {}^{u+2}C_4\delta^4 y_0 + ... + {}^{u+k-1}C_{2k-1}\delta^{2k-1} y_{-1/2} + {}^{u+k}C_{2k}\delta^{2k} y_0 + ... \end{aligned}$$

That is, we have,

$$y_u = y_0 + {}^uC_1\delta y_{-1/2} + {}^{u+1}C_2\delta^2 y_0 + {}^{u+1}C_3\delta^3 y_{-1/2} + {}^{u+2}C_4\delta^4 y_0 + ... + {}^{u+k-1}C_{2k-1}\delta^{2k-1} y_{-1/2} + {}^{u+k}C_{2k}\delta^{2k} y_0 + ...$$

This is Gauss's backward interpolation formula for equal intervals. This formula employs the odd differences just above the central line through y_0 and the even differences on the central line.

Note: In Gauss's interpolation formula, we observe that

(i) The indices of the operator Δ in the terms are in ascending order of units starting from 0.

(ii) The suffixes of the entries, i.e. of *y*, appear as 0 and -1/2 in alternate terms.

(iii) $T_{2k} = {}^{u+k-1}C_{2k-1}\delta^{2k-1} y_{-1/2},\ \ T_{2k+1} = {}^{u+k}C_{2k}\delta^{2k} y_0 \ \ for\, k \geq 1$

4.04.03: Stirling's interpolation formula:

Gauss's forward formula is-

$$y_u = y_0 + {}^uC_1\Delta y_0 + {}^uC_2\Delta^2 y_{-1} + {}^{u+1}C_3\Delta^3 y_{-1} + {}^{u+1}C_4\Delta^4 y_{-2} + ... + {}^{u+k-1}C_{2k-1}\Delta^{2k-1} y_{-k+1} + {}^{u+k-1}C_{2k}\Delta^{2k} y_{-k} + ...$$

...……….(1)

And Gauss's backward formula with forward difference operator Δ is-

$$y_u = y_0 + {}^uC_1\Delta y_{-1} + {}^{u+1}C_2\Delta^2 y_{-1} + {}^{u+1}C_3\Delta^3 y_{-2} + {}^{u+2}C_4\Delta^4 y_{-2} + ... + {}^{u+k-1}C_{2k-1}\Delta^{2k-1} y_{-k} + {}^{u+k}C_{2k}\Delta^{2k} y_{-k} + ... + ...$$

…………(2)

Taking the mean of (1) and (2), we get,

$$y_u == y_0 + u.\frac{\Delta y_0 + \Delta y_{-1}}{2} + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}.\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + ...$$

$$+ \frac{u(u^2-1^2)(u^2-2^2)(u^2-3^2).....[u^2-(k-1)^2]}{(2k-1)!}.\frac{\Delta^{2k-1} y_{-k+1} + \Delta^{2k-1} y_{-k}}{2}$$

$$+ \frac{u^2(u^2-1^2)(u^2-2^2)(u^2-3^2).....[u^2-(k-1)^2]}{(2k)!}.\Delta^{2k} y_{-k} + ...$$

$$\Rightarrow y_u = y_0 + u.\mu.\delta y_0 + \frac{u^2}{2!}\delta^2 y_0 + {}^{u+1}C_3.\mu\delta^3 y_0 + \frac{u^2(u^2-1^2)}{4!}\delta^4 y_0 +.{}^{u+2}C_5\mu\delta^5 y_0$$

$$+ \frac{u^2(u^2-1^2)(u^2-2^2)}{6!}.\delta^6 y_0 + {}^{u+3}C_7\mu\delta^7 y_0 + \frac{u^2(u^2-1^2)(u^2-2^2)(u^2-3^2)}{8!}.\delta^8 y_0 + ...$$

where $u = (x - x_0)/h$

This is Stirling's interpolation formula. This formula employs the mean of the odd differences above and below the central line and even differences on the central line.

4.04.04 Bessel's interpolation formula:

Gauss's forward interpolation formula is-

$$y_u = y_0 + {}^uC_1\Delta y_0 + {}^uC_2\Delta^2 y_{-1} + {}^{u+1}C_3\Delta^3 y_{-1} + {}^{u+1}C_4\Delta^4 y_{-2} + ... + {}^{u+k-1}C_{2k-1}\Delta^{2k-1} y_{-k+1} + {}^{u+k-1}C_{2k}\Delta^{2k} y_{-k} + ...$$

....……….(1)

And Gauss's backward interpolation formula with forward difference operator Δ is-

$$y_u = y_0 + {}^uC_1\Delta y_{-1} + {}^{u+1}C_2\Delta^2 y_{-1} + {}^{u+1}C_3\Delta^3 y_{-2} + {}^{u+2}C_4\Delta^4 y_{-2} + ... + {}^{u+k-1}C_{2k-1}\Delta^{2k-1} y_{-k} + {}^{u+k}C_{2k}\Delta^{2k} y_{-k} + ... + ...$$

If we transfer the origin in the above formula to unity, then, we get,

$$y_u = y_1 + {}^{u-1}C_1\Delta y_0 + {}^uC_2\Delta^2 y_0 + {}^uC_3\Delta^3 y_{-1} + {}^{u+1}C_4\Delta^4 y_{-1} + ... + {}^{u+k-2}C_{2k-1}\Delta^{2k-1} y_{-k+1} + {}^{u+k-1}C_{2k}\Delta^{2k} y_{-k+1}. + ...$$

……….(2)

Taking the mean of (1) and (2), we get,

$$y_u = \frac{1}{2}(y_0 + y_1) + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!} \times \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(u-1/2)u(u-1)}{3!}\Delta^3 y_{-1}.$$

$$+ \frac{(u+1)u(u-1)(u-2)}{4!} \times \frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2} + ... + \frac{(u+k-1)(u+k-2)...(u-k)}{(2k)!} \times \frac{\Delta^{2k} y_{-k} + \Delta^{2k} y_{-k+1}}{2}$$

$$+ \frac{(u-1/2)(u+k-1)(u+k-2).....(u-k)}{(2k+1)!}.\Delta^{2k+1} y_{-k} + ...$$

This is Bessel's interpolation formula. In this formula, the coefficients of the odd order differences are all zero at $u = 0, 1/2, 1$. This is the advantage of Bessel's formula.

4.04.05. Another form of Bessel's formula:

Bessel's formula can be made more symmetric by putting $u = v + 1/2$ and then, we get,

$$y_u = y_{v+1/2} = \frac{1}{2}(y_0 + y_1) + v.\Delta y_0 + \frac{v^2 - 1/4}{2!} \times \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{v(v^2 - 1/4)}{3!}\Delta^3 y_{-1}.$$
$$+ \frac{(v^2 - 1/4)(v^2 - 9/4)}{4!} \times \frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \frac{(v^2 - 1/4)(v^2 - 9/4)\ldots(v^2 - (2k-1)^2/4)}{(2k)!} \times \frac{\Delta^{2k} y_{-k} + \Delta^{2k} y_{-k+1}}{2}$$
$$+ \frac{v(v^2 - 1/4)(v^2 - 9/4)\ldots(v^2 - (2k-1)^2/4)}{(2k+1)!}.\Delta^{2k+1} y_{-k} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

This is an important form of Bessel's formula in application viewpoint.

Corollary: If we put $u = 1/2 \;\; or \;\; v = 0,$ then Bessel's formula takes the form,

$$y_{1/2} = \frac{1}{2}(y_0 + y_1) - \frac{1}{8} \times \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{3}{128} \times \frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ (-1)^k \frac{[1.3.5\ldots\ldots(2k-1)]^2}{2^{2k}.(2k)!}. \times \frac{\Delta^{2k} y_{-k} + \Delta^{2k} y_{-k+1}}{2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

This formula is called the *formula for interpolating to halves* or *formula for halving an interval*. It is used for computing values of the function midway between any two given values. This can also be written by using the operators μ, δ. Then its form is,

$$y_{1/2} = \mu\, y_{1/2} - \frac{1}{8} \times \mu\,\delta^2 y_{1/2} + \frac{3}{128} \times \mu\,\delta^4 y_{1/2} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$