Mathematics Major Notes

Class: B.Sc. 1st Semester

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UNIT 1: Relations, Equivalence relations, mappings, binary compositions **Marks**: 10

TOPIC: Relations, Equivalence relations

PREREQUISITES: In subsections 1.1 and 1.2, we recall some points which are very much needed for our future discussions.

1.1: Ordered sets, Ordered pairs, ordered triples and n-tuples

An *ordered set* is nothing but a particular set whose elements maintain some order. This is distinguished from other ordinary sets only in the basis of order of the elements. If the order of the elements is changed, we get a different ordered set. To distinguish ordered sets from unordered ordinary sets, we normally use the bracket ( ) instead of {} to represent ordered sets. For example, {1, 2, 3, 4} is an ordinary set while (1, 2, 3, 4) is an ordered set.

An ordered set formed by two elements is called an ordered pair. For example, (2, 5) is an ordered pair. Similarly, an ordered set consisting of three elements is an ordered triple and in general, an ordered set of n elements is an n-tuple.

1.2: Cartesian product of two sets

If A and B are two non-empty sets, then their Cartesian product is denoted by $A \times B$ and is defined by

$$A \times B = \{ (a,b) : a \in A \text{ and } b \in B \}$$

1.3: RELATIONS

If A and B are two non-empty sets, then any subset of A × B is called a relation from A to B. That is, if R ⊆ A × B, then R is a relation from A to B.

Relations are generally denoted by R, R_1, R_2 etc. If R is a relation and $(a,b) \in R$, then we denote this fact by the notation aRb also and we call it as "a is R-related to b". Thus aRb and $(a,b) \in R$ imply the same.

1.3.01: Domain and Range of a relation: The domain and range of a relation R are respectively denoted by D and E and are defined as follows:

$$D=\{a\in A : aRb \quad for\ some\ b\in B\},\quad E=\{b\in B: aRb\ for\ some\ a\in A\}$$

Clearly, $D\subseteq A$ and $E\subseteq B$.

1.3.02: Total number of distinct relations from a set to another set

Suppose A and B be two non-empty sets. Then the total number of distinct elements in A×B is

n (A).n (B) and so we have $2^{n(A).n(B)}$ numbers of distinct subsets of A×B. Since, by definition, every subset of A×B is a relation from A to B, so it clearly follows that there are $2^{n(A).n(B)}$ distinct relations from A to B. From this we have that the total number of distinct relations from a finite non-empty set to another finite non-empty set is always a power of two and hence an even number.

Ex. 1.3.02.01: If A= {a, b, c} and B= {1, 2, 3, 4 }, then the following are some examples of relations from A to B:

(i) R_1 ={(a,2), (c,1), (b,4),(b,1), (c,2),(b,2)}

(ii) R_2 ={(a,4), (c,2), (b,1),(b,4), (c,1),(b,2),(a,2), (a,1), (c,3)}

(iii) R_1 ={(a,1), (a,2), (a,3), (a,4), (b,1), (b,2),(b,3), (c,1),(c,2)}

1.4: The void and the universal relations from a set A to a set B

A relation R from a set A to a set B is called

(i) The void relation from A to B if $R=\phi$

(ii) The universal relation from A to B if $R=A\times B$

1.5: Relations in a set

If $R\subseteq A\times A$, then R is a relation from A to A. In this case we generally say that R is a relation in the set A. This is because of the fact that the relation R relates among the elements within the set A.

Ex.1.5.01: Let L be the set of all lines on the xy-plane. Then the following are some good examples of relations in the set L:

(i) $R_1=\{(l_1,l_2)\in L\times L : l_1\perp l_2\ i.e.\ the\ line l_1\ is\ perpendicular to the line l_2\}$

(ii) $R_2=\{(l_1,l_2)\in L\times L:\ the\ line l_1\ is\ parallel to the line l_2\}$

(iii) $R_3=\{(l_1,l_2)\in L\times L:\ the\ line l_1\ cuts the line l_2\}$

Ex.1.5.02: Let T be the set of all triangles on the xy-plane. Then the following are some examples of relations in T:

(i) $R_4 = \{(\Delta_1, \Delta_2) \in T \times T : \Delta_1 \cong \Delta_2 \text{ } i.e. \text{ } the\ triangle \Delta_1\ is congruent to the\ triangle\ \Delta_2\}$

(ii) $R_5 = \{(\Delta_1, \Delta_2) \in T \times T : \Delta_1 \approx \Delta_2 \text{ } i.e. \text{ } the\ triangle \Delta_1\ is similar to the\ triangle\ \Delta_2\}$

(iii) $R_6 = \{(\Delta_1, \Delta_2) \in T \times T : area(\Delta_1) = area(\Delta_2) \text{ } i.e. \text{ } the\ area of\ \Delta_1\ is equal to the area of\ \Delta_2\}$

Ex.1.5.04: Let Z be the set of all integers. Then the following are some relations in Z:

(i) $R_7 = \{(m,n) \in \mathrm{Z} \times \mathrm{Z} : m \mid n \text{ } i.e. \text{ } m\ divides n\}$

(ii) $R_8 = \{(m,n) \in \mathrm{Z} \times \mathrm{Z} : m \equiv n \pmod p \text{ } where\ p(>1) \in \mathrm{N} \text{ } i.e., p \mid (m-n)\}$

(iii) $R_9 = \{(m,n) \in \mathrm{Z} \times \mathrm{Z} : m \geq n\}$

(iv) $R_{10} = \{(m,n) \in \mathrm{Z} \times \mathrm{Z} : m^2 > 2n+1\}$

Ex.1.5.05: For the set N of all natural numbers, define R_{11}, R_{12} in $\mathrm{N} \times \mathrm{N}$ as follows:

(i) $R_{11} = \{(a,b),(c,d) \in \mathrm{N} \times \mathrm{N} : ad = bc\}$

(ii) $R_{12} = \{(a,b),(c,d) \in \mathrm{N} \times \mathrm{N} : ab = cd\}$

Then R_{11}, R_{12} are relations in $\mathrm{N} \times \mathrm{N}$.

Ex.1.5.06: For any non-empty set X, the relation *R* is defined in $\mathrm{X} \times \mathrm{X}$ as

$R = \{(x,x): \ x \in \mathrm{X}\}$ is called the ***identity relation*** in X.

1.6: Different types of relations in a set:

A relation *R* in a non-empty set A (i.e., $R \subseteq \mathrm{A} \times \mathrm{A}$) is called

(i) Reflexive iff $a \in A \Rightarrow a\,Ra$

(ii) Symmetric iff $aRb \Rightarrow bRa$

(iii) Anti-symmetric iff $aRb, bRa \Rightarrow a = b$

(iv) Transitive iff $aRb,\ bRc \Rightarrow aRc$

1.7: Two important relations in a set

We have two more special types of relations in a non-empty set which has some algebraic as well as analytical importance. These are used, in a certain sense, as tools to characterize and analyze internal structure of sets.

A relation *R* in a set A is said to be

(i) An **equivalence relation** if it is **reflexive, symmetric** and **transitive**

(ii) A **partial order relation** if it **is reflexive, anti-symmetric** and **transitive**

We mainly study equivalence relations in detail in this discourse.

1.8: Some examples and counterexamples of equivalence relations

Ex.1.8.01: Consider the set L of all the lines on the xy-plane**.** Then the relation R in L defined by $R=\{(l_1,l_2)\in L\times L:\ the\ line\ l_1\ is\ parallel\ to\ the\ line\ l_2\}$ is clearly an equivalence relation while $R'=\{(l_1,l_2)\in L\times L:\ l_1\perp l_2\ i.e., the\ line\ l_1\ is\ perpendicular\ to\ the\ line\ l_2\}$ is not an equivalence relation in L. For, in case of R' we can easily see that the transitivity does not hold.

Ex.1.8.02: Consider the set T of all the triangles on the xy-plane**.** In this case the relations in T defined by $R_4=\{(\Delta_1,\Delta_2)\in T\times T:\Delta_1\cong\Delta_2\ i.e.\ the\ triangle\ \Delta_1\ is\ congruent\ to\ the\ triangle\ \Delta_2\}$ and $R_5=\{(\Delta_1,\Delta_2)\in T\times T:\Delta_1\approx\Delta_2\ i.e.\ the\ triangle\ \Delta_1\ is\ similar\ to\ the\ triangle\ \Delta_2\}$ are equivalence relations.

Ex.1.8.03: In the set Z of all integers, the relation $R=\{(m,n)\in Z\times Z: m\,|\,n\ i.e.\ m\ divides\ n\}$ is not an equivalence relation. For, this relation is not symmetric. For example, 3|6 and hence by definition, 3*R*6; but $6\nmid 3$ and hence $6\not R 3$. So, R is not symmetric.

Ex.1.8.04: In Z, the relation R defined by

$R=\{(m,n)\in Z\times Z: m\equiv n\ (\mathrm{mod}\,p)\ where\ p(>1)\in N\ i.e., p\,|\,(m-n)\}$ is an equivalence relation. For,

(i) **Reflexivity:**

We have**,**

$$\begin{aligned} n\in Z &\Rightarrow n-n=0=0.p,\ a\ multiple\ of\ p\\ &\Rightarrow p\,|\,(n-n)\\ &\Rightarrow n\equiv n\ (\mathrm{mod}\,p)\\ &\Rightarrow nRn,\ \forall n\in Z \end{aligned}$$

Therefore, R is reflexive.

(ii) **Symmetry**: We have**,**

$$\begin{aligned} mRn &\Rightarrow p\,|\,(m-n)\\ &\Rightarrow p\,|-(m-n) \quad [by\ definition\ of\ divisibility]\\ &\Rightarrow p\,|\,(n-m)\\ &\Rightarrow n\equiv m\ (\mathrm{mod}\,p)\\ &\Rightarrow nRm \quad [by\ definition\ of\ R] \end{aligned}$$

Therefore, R is symmetric.

(iii) **Transitivity**: We have,

$$\begin{aligned} m\,R\,n,\ n\,R\,l &\Rightarrow p \mid (m-n),\ p \mid (n-l) \\ &\Rightarrow p \mid (m-n)+(n-l) \quad [by\ property\ of\ divisibility] \\ &\Rightarrow p \mid (m-l) \\ &\Rightarrow m \equiv l \ (\text{mod}\, p) \\ &\Rightarrow mRl \qquad [by\ definition\ of\ R] \end{aligned}$$

Therefore, *R* is transitive and hence we can ultimately conclude that *R* is an equivalence relation in Z.

Ex.1.8.05: Consider the relation $R = \{(a,b),(c,d) \in \mathrm{N}\times\mathrm{N} : ad = bc\}$ in N. Here,

(i) **Reflexivity**: We have,

$$\begin{aligned} (a,b) \in \mathrm{N}\times\mathrm{N} &\Rightarrow a,b \in \mathrm{N} \\ &\Rightarrow ab = ba \quad [\because multiplication\ of\ reals\ is\ commutative] \\ &\Rightarrow (a,b)\ R\ (a,b) \qquad [by\ definition\ of\ R] \end{aligned}$$

Therefore R is reflexive.

(ii) **Symmetry**: We have,

$$\begin{aligned} (a,b)\ \ R\ (c,d) &\Rightarrow ad = bc \quad [by\ definition\ of\ R] \\ &\Rightarrow bc = ad \\ &\Rightarrow cb = da \quad [by\ commutativity\ of\ multiplication] \\ &\Rightarrow (c,d)\ R\ (a,b) \qquad [by\ definition\ of\ R] \end{aligned}$$

Therefore R is symmetric**.**

(iii) **Transitivity**: We have,

$$\begin{aligned} (a,b)\ \ R\ (c,d),\ (c,d)\ R\ (e,f) &\Rightarrow ad = bc\ ,\ cf = de \quad [by\ definition\ of\ R] \\ &\Rightarrow adf = bcf\ ,\ bcf = bde \\ &\Rightarrow adf = bde \\ &\Rightarrow af = be \\ &\Rightarrow (a,b)\ R\ (e,f) \qquad [by\ definition\ of\ R] \end{aligned}$$

So, *R* is transitive and hence it follows that *R* is an equivalence relation.

1.9: **Equivalence classes**:

Suppose *R* is an equivalence relation in a non-empty set *A*. Then it gives rise to some subsets of *A* which are called equivalence sets or equivalence classes. The formal definition of the equivalence classes is as follows:

The equivalence class of $a \in A$ is denoted by A_a or by $[a]$ or by $\bar{a}$ and is defined by

$$[a]=\{x\in A:(x,a)\in R \ i.e., \ x \ R \ a\}$$

That is, $[a]$ is the subset of *A* that contains all those elements in *A* which are *R*-related to *a*.

Ex.1.9.01: Consider the set Z of all integers and the relation *R* in Z defined as follows:

$R=\{(m,n):(m,n)\in \mathrm{Z}\times\mathrm{Z} \ \ and \ \ m\equiv n \ (\mathrm{mod}5)\}$. Further consider the elements $0,1,2,3,4\in \mathrm{Z}$.

Here, we first find the elements in $[0]$.

We have,

$$\begin{aligned}[0]=\{x\in \mathrm{Z}:0Rx\}=\{x\in \mathrm{Z}:x\equiv 0 \ (\mathrm{mod}5\}=\{x\in \mathrm{Z}:5\,|\,(x-0)\} \\ =\{x\in \mathrm{Z}:5\,|\,x\} \\ =\{x\in \mathrm{Z}:x=5m, \ m\in \mathrm{Z}\} \\ =\{0, \pm 5, \pm 10, \pm 15, \pm 20, \pm 25,.....\}\end{aligned}$$

$$\begin{aligned}[1]=\{x\in \mathrm{Z}:1Rx\}=\{x\in \mathrm{Z}:x\equiv 1(\mathrm{mod}5\}=\{x\in \mathrm{Z}:5\,|\,(x-1)\} \\ =\{x\in \mathrm{Z}:x-1=5m, \ m\in \mathrm{Z}\} \\ =\{x\in \mathrm{Z}:x=5m+1, \ m\in \mathrm{Z}\} \\ =\{.....,-14,-9,-4, 1, 6, 11, 16,.....\}\end{aligned}$$

$$\begin{aligned}[2]=\{x\in \mathrm{Z}:2Rx\}=\{x\in \mathrm{Z}:x\equiv 2(\mathrm{mod}5\}=\{x\in \mathrm{Z}:5\,|\,(x-2)\} \\ =\{x\in \mathrm{Z}:x-2=5m, \ m\in \mathrm{Z}\} \\ =\{x\in \mathrm{Z}:x=5m+2, \ m\in \mathrm{Z}\} \\ =\{.....,-13,-8,-3, 2, 7, 12, 17,.....\}\end{aligned}$$

Similarly, $[3]=\{.....,-12,-7,-2, 3, 8, 13, 18, ...\}$, $\quad [4]=\{....,-11,-6,-1,4,9,14,19,.........\}$

Further we see that

$$\begin{aligned}&[0]=[\pm 5]=[\pm 10]=[\pm 15]=[\pm 20]=[\pm 25]=.......\\&[1]=[-4]=[6]=[-9]=[11]=[-14]=[16]=.......\\&[2]=[-3]=[7]=[-8]=[12]=[-13]=[17]=.......\\&[3]=[-2]=[8]=[-7]=[13]=[-12]=[18]=.......\\&[4]=[-1]=[9]=[-6]=[14]=[-11]=[19]=.......\end{aligned}$$

That is, we have only five distinct equivalence classes in *A* determined by the relation *R*.

Note: Equivalence classes in a set have some nice properties. These properties give rise to a partition [Definition given below] of the set A.

1.10: Properties of equivalence classes:

Theorem 1.10.01: If R is an equivalence relation in a set A and $a, b \in A$, then

(i) $a \in [a]$

(ii) $b \in [a] \Rightarrow [a] = [b]$

(iii) $[a] = [b] \Leftrightarrow aRb$

(iv) Either $[a] = [b]$ or $[a] \cap [b] = \phi$

Proof:

(i) We have,

R is an equivalence relation in A and $a \in A$

$\Rightarrow a \ R \ a \qquad [\because R \text{ is reflexive}]$

$\Rightarrow a \in [a] \qquad [by\ definition\ of\ equivalence\ classes]$

(ii) We have,

$b \in [a] \Rightarrow b \ R \ a \ \ and \ \ a \ R \ b \ \ [by\ symmetry\ of\ R]$

Now we show that $[a] = [b]$. For this we show that $[a] \subseteq [b]$ & $[b] \subseteq [a]$.

We get,

$x \in [a],\ a \ R \ b \Rightarrow x \ R \ a,\ a \ R \ b$

$\Rightarrow x \ R \ b \qquad [by\ transitivity\ of\ R]$

$\Rightarrow x \in [b] \qquad [by\ definition\ of\ equivalence\ classes]$

Therefore, it follows that $[a] \subseteq [b]$

Again,

$y \in [b],\ b \ R \ a \Rightarrow y \ R \ b,\ b R \ a$

$\Rightarrow y \ R \ a \qquad [by\ transitivity\ of\ R]$

$\Rightarrow x \in [a] \qquad [by\ definition\ of\ equivalence\ classes]$

Therefore, it follows that $[b] \subseteq [a]$

Finally $[a] \subseteq [b]$ & $[b] \subseteq [a] \Rightarrow [a] = [b]$

(iii) Let us first assume that $[a] = [b]$

Then from (i), we have, $a \in [a]$

Now,

$a \in [a] \Rightarrow a \in [b] \quad [\because [a] = [b]]$

$\Rightarrow a \ R \ b \qquad [by\ definition]$

Conversely, let $a \ R \ b$. Then, by definition, $a \in [b]$ and hence by (ii), $[a] = [b]$.

(iv) For two equivalence classes $[a]$ and $[b]$, we have two cases:

Case 1: $[a] \cap [b] = \phi$ or **Case 2**: $[a] \cap [b] \neq \phi$

For case 1, we are nothing to do.

If $[a] \cap [b] \neq \phi$, then $\exists \ x \in [a] \cap [b]$ and hence

$$x \in [a] \cap [b] \Rightarrow x \in [a], \ x \in [b]$$
$$\Rightarrow [a] = [x], \ [b] = [x]$$
$$\Rightarrow [a] = [x] = [b]$$
$$\Rightarrow [a] = [b]$$

Thus $[a] \cap [b] \neq \phi \Rightarrow [a] = [b]$

Hence either $[a] \cap [b] = \phi$ *or* $[a] = [b]$

1.11: Partitions:

The term partition comes from the word "part". When a set is divided into some parts under certain conditions, then we get a partition of the set. More precisely, by a partition of a non-empty set A we mean a division of the set A into mutually disjoint subsets of it.

Formally, a collection $\mathrm{P} = \{A_1, A_2, A_3, \ldots\ldots, A_n\}$ of subsets of a non-empty set A is called a partition if

(i) $A_i \neq \phi, \ \forall i = 1,2,3,\ldots, n$

(ii) $A_i \cap A_j = \phi, \forall i \neq j$

(iii) $\bigcup_{i=1}^{n} A_i = A$

For example $\mathrm{P} = \{ [0], \ [1], [2], [3], [4] \}$ is a partition of Z,

Theorem 1.11.01: [Fundamental theorem of equivalence relations]

Every equivalence relation in a non-empty set A induces/ determines a partition of A and conversely, a partition of A defines in an equivalence relation in A.

Proof: Part I: Let us first assume that R is an equivalence relation in a non-empty set A. We show that R determines a partition of A.

Let $\mathrm{P} = \{ [a] : a \in \mathrm{A}, \ [a] \ \textit{is an equivalence class of} \ \mathrm{A} \ \textit{given by the equivalence relation} \ R \}$ be the collection of all the distinct equivalence classes of A determined by R. Then, by properties of equivalence classes, we have,

(i) $a \in \mathrm{A} \Rightarrow a \in [a]$

Now, $a \in [a] \Rightarrow [a] \neq \phi, \ \ \forall\ [a] \in \mathrm{P}$

(ii) For two equivalence classes $[a]$ and $[b]$, we have, either $[a] \cap [b] = \phi$ or $[a] = [b]$

So, distinct members of P are pairwise disjoint.

(iii) We have that $\bigcup_{[a]\in \mathrm{P}} [a] = \mathrm{A}$

For, we have,

$$a \in \mathrm{A} \Rightarrow a \in [a] \subseteq \bigcup_{[a]\in \mathrm{P}} [a]$$

$$\therefore \mathrm{A} \subseteq \bigcup_{[a]\in \mathrm{P}} [a] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Conversely,

$$x \in \bigcup_{[a]\in \mathrm{P}} [a] \Rightarrow x \in [a] \quad \textit{for some}\ [a] \in \mathrm{P}$$

$$\Rightarrow x \in [a] \subseteq \mathrm{A} \qquad [\because [a] \subseteq \mathrm{A}]$$

$$\Rightarrow x \in \mathrm{A}$$

$$\therefore \bigcup_{[a]\in \mathrm{P}} [a] \subseteq \mathrm{A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now, (1) & (2) $\Rightarrow \bigcup_{[a]\in \mathrm{P}} [a] = \mathrm{A}$

(i), (ii) & (iii) imply that P is partition of A. We have seen that this partition is determined by the equivalence relation R. So, we can conclude that R induces the partition P.

Part II: Let $\mathrm{P} = \{\mathrm{A}_1, \ \mathrm{A}_2, \ \mathrm{A}_3, \ldots\ldots, \ \mathrm{A}_n\}$ be a partition of the non-empty set A so that

(i) $A_i \neq \phi, \ \forall i = 1,2,3,\ldots, n$

(ii) $A_i \cap A_j = \phi, \forall i \neq j$

(iii) $\bigcup_{i=1}^{n} A_i = A$

Now, let us define a relation R in A by

$$a\ R\ b \Leftrightarrow a,b \in \mathrm{A}_i \ \ \textit{for some}\ \ \mathrm{A}_i \in \mathrm{P}$$

We now show that R is an equivalence relation in A.

(1) Reflexivity: We have,

$$a \in \mathrm{A} \Rightarrow a \in \bigcup_{i=1}^{n} \mathrm{A}_i \Rightarrow a \in \mathrm{A}_i \quad \textit{for some } i = 1,2,\ldots,n$$

$$\Rightarrow a \in \mathrm{A}_i \,, a \in \mathrm{A}_i \;\; [\because$$

$$\Rightarrow a \; R \; a$$