METRIC SPACES

Observations: While the concept of distance is concerned, we observe the following facts:

i) To define distance we need two points or objects in a set.
ii) The distance between two points is always unique and a non-negative real number.
iii) The distance from one point to itself is always zero.
iv) The distance from one point to another point is always equal to the distance from the second point to the first point.
v) If three points are given, then the distance between any two of them never exceeds the sum of the distances between 1st point and 2nd point, and 2nd point and 3rd point.

These are quite common facts seen in one, two and three dimensional spaces. On generalization of these facts/results, mathematicians introduced a very general kind of concept about distances between points in an arbitrary set, called metric, which has been defined as follows:

Metric or a distance function:

For a non-empty set X, a mapping $d: X \times X \to R$, where R is the set of real numbers, is called a *metric* or a *distance function* for X if the following four conditions, called metric conditions, are satisfied:

M_1) $d(x, y) \geq 0, \forall x, y \in X$

M_2) $d(x, y) = 0 \Leftrightarrow x = y$

M_3) $d(x, y) = d(y, x), \forall x, y \in X$ [*Symmetry*]

M_4) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$ [*Triangular Inequality*]

Metric spaces

If d is a metric for a non-empty set X, then the pair (X, d) is called a metric space. That is, a non-empty set together with a metric for it is a metric space.

Pseudo-metric

For a non-empty set X, a mapping $d: X \times X \to R$, where R is the set of real numbers, is called a pseudo-metric or semi-metric for X if the following four conditions are satisfied:

m_1) $d(x, y) \geq 0, \forall x, y \in X$

m_2) $x = y \Longrightarrow d(x, y) = 0$

m_3) $d(x, y) = d(y, x), \forall x, y \in X$ [*Symmetry*]

m_4) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$ [*Triangular Inequality*]

Note: For a pseudo-metric, we may have that $d(x, y) = 0$ even if $x \neq y$. That is, for a pseudo-metric,$x = y \Longrightarrow d(x, y) = 0$, but not conversely. So, it clear follows that every metric is a pseudo-metric, but a pseudo-metric is not necessarily a metric.

Examples of pseudo-metrics which are not metrics

Example 1: Let R be the set of all real numbers and $d: R \times R \rightarrow R$ such that$d(x, y) = |x^2 - y^2|$, for all$x, y \in R$. Then, d is a pseudo-metric which is not a metric.

For, m$_1$) $x, y \in R \Longrightarrow x^2 - y^2 \in R \Longrightarrow |x^2 - y^2| \geq 0 \Longrightarrow d(x, y) \geq 0$

m$_2$) $x = y \Longrightarrow x^2 = y^2 \Longrightarrow |x^2 - y^2| = 0 \Longrightarrow d(x, y) = 0$

But for$k > 0$, $k \neq -k$ and $d(k, -k) = |k^2 - (-k)^2| = |k^2 - k^2| = 0$

That is, we have, $d(x, y) = 0$ even if $x \neq y$.

m$_3$) $d(x, y) = |x^2 - y^2| = |-(y^2 - x^2)| = |(y^2 - x^2)| = d(y, x), \forall x, y \in R$

m$_4$) $d(x, y) = |x^2 - y^2| = |x^2 - z^2 + z^2 - y^2| \leq |x^2 - z^2| + |z^2 - y^2|$

$$= d(x, z) + d(z, y), \forall x, y, z \in X$$

All the above ensure us that d is a pseudo-metric, but not a metric.

Example 2: Let c be the set of all convergent real sequences and $d: c \times c \rightarrow R$ such that$d(\langle x_n \rangle, \langle y_n \rangle) = |x - y|$, where x *and* y are the limits of the sequences $\langle x_n \rangle$ *and* $\langle y_n \rangle$ respectively. Then, d is a pseudo-metric, but not a metric.

Here, m$_1$, m$_3$ and m$_4$ can easily be established as usual ways.

For m$_2$, let, $x = \langle x_n \rangle$ *and* $y = \langle y_n \rangle$ are two convergent real sequences in c converging to the same limit k. It is always possible, because if we may have many real sequences which converge to the same limit. For example, $\langle \frac{1}{n} \rangle, \langle \frac{1}{2^n} \rangle$, $\langle \frac{1}{3^n} \rangle$, $\langle \frac{1}{5^n} \rangle$ are some distinct sequences which converge to the same limit 0.

In the above case, $d(x, y) = d(\langle x_n \rangle, \langle y_n \rangle) = |k - k| = 0$, but $x = \langle x_n \rangle \neq \langle y_n \rangle = y$.

Example 3: Let R be an equivalence relation on a non-empty set S and X be the set of all equivalence classes in S determined by the equivalence relation R in S. Then $d: S \times S \rightarrow R$ such that

$$d(x, y) = \begin{cases} 0, & if\ x\ and\ y\ belong\ to\ the\ same\ equivalence\ class\ in\ X \\ 1, & if\ x\ , y\ belong\ to\ different\ equivalence\ classes\ in\ X \end{cases}$$

Then, d is a pseudo-metric for S.

§ *Some important inequalities useful to establish the 4^th^ metric condition M₄*

1. For any $z_1, z_2, \ldots, z_n \in R \text{ or } C$, we have,
   i) $|z_1 + z_2| \le |z_1| + |z_2|$
   ii) $\frac{|z_1+z_2|}{1+|z_1+z_2|} \le \frac{|z_1|}{1+|z_1|} + \frac{|z_2|}{1+|z_2|}$
   iii) $|z_1 + z_2 + z_3 + \cdots + z_n| \le |z_1| + |z_2| + |z_3| + \cdots + |z_n|$

2. **Minkowski's inequality**
   i) If $p \ge 1 \text{ and } a_i, b_i \ge 0, i = 1,2,3, \ldots, n, \text{then},$

$$\left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}$$

   ii) If $p \ge 1 \text{ and } a_i, b_i \text{ are complex numbers}, i = 1,2,3, \ldots, n, \text{then},$

$$\left(\sum_{i=1}^{n}|a_i + b_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n}|a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}|b_i|^p\right)^{\frac{1}{p}}$$

   iii) If $0 < p < 1 \text{ and } a_i, b_i \ge 0, i = 1,2,3, \ldots, n, \text{then},$

$$\sum_{i=1}^{n}(a_i + b_i)^p \le \sum_{i=1}^{n} a_i^p + \sum_{i=1}^{n} b_i^p$$

   iv) If $0 < p < 1 \text{ and } a_i, b_i \text{ are complex numbers}, i = 1,2,3, \ldots, n, \text{then},$

$$\sum_{i=1}^{n}|a_i + b_i|^p \le \sum_{i=1}^{n}|a_i|^p + \sum_{i=1}^{n}|b_i|^p$$

3. **Holder's inequality**
   If $p > 1, \frac{1}{p} + \frac{1}{q} = 1 \text{ and } a_i, b_i \ge 0, i = 1,2,3, \ldots, n, \text{then},$

$$\sum_{i=1}^{n} a_i\, b_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} . \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}$$

4. **Cauchy-Schwarz inequality**
   If $a_i, b_i \text{ are complex numbers}, i = 1,2,3, \ldots, n, \text{then},$

$$\sum_{i=1}^{n}|a_i\, b_i| \le \left(\sum_{i=1}^{n}|a_i|^2\right)^{\frac{1}{2}} . \left(\sum_{i=1}^{n}|b_i|^2\right)^{\frac{1}{2}}$$

§ Some examples of metric spaces

Example1: Consider the set R of real numbers and $d: R \times R \to R$ such that $d(x, y) = |x - y|$, for all $x, y \in R$. Then d is a metric for R. This metric is known as the usual metric for R.

Solution:

M_1) We have,

$$x, y \in R \Longrightarrow x - y \in R$$

$$\Longrightarrow |x - y| \geq 0$$

$$\Longrightarrow d(x, y) \geq 0 \ \text{[By definition of d]}$$

Hence, M_1 is satisfied.

M_2) We have,

$$d(x, y) = 0 \Leftrightarrow |x - y| = 0$$

$$\Leftrightarrow x - y = 0 \ [\because |a| = 0 \Longrightarrow a = 0]$$

$$\Leftrightarrow x = y$$

Hence, M_2 is satisfied.

M_3) We have,

$$d(x, y) = |x - y| = |-(y - x)|$$

$$= |y - x| \quad [\because |-a| = |a|, \forall a \in R]$$

$$= d(y, x), \forall x, y \in R \quad \text{[By definition of d]}$$

Hence, M_3 is satisfied.

M_4). For $x, y, z \in R$, we have,

$$d(x, y) = |x - y| = |(x - z) + (z - y)|$$

$$\leq |x - z| + |z - y| \quad [\because |a + b| \leq |a| + |b|, \forall a, b \in R]$$

$$= d(x, z) + d(z, y), \ \ \forall x, y, z \in R$$

Hence, M_4 is satisfied.

Thus all the metric conditions are satisfied by d and hence d is a metric for R and consequently (R, d=|. |) is a metric space.

Example2: Consider an arbitrary non-empty set X and $d: X \times X \to R$ such that

$$d(x, y) = \begin{cases} 0, & if\ x = y \\ 1, & if\ x \neq y \end{cases}$$

Then d is a metric for X, called the discrete metric for X.

Solution:

M_1) We have,

$$x, y \in X \Longrightarrow x = y \text{ or } x \neq y$$

$$\Longrightarrow d(x, y) = 0 \text{ or } d(x, y) = 1$$

$$\Longrightarrow d(x, y) \geq 0$$

Hence, M_1 is satisfied.

M_2) By definition, we have,

$$x = y \Longrightarrow d(x, y) = 0$$

On the other hand, when $d(x, y) = 0$, then, also we must have that $x = y$. For, otherwise, we have that $d(x, y) = 1$.

So, it follows that $d(x, y) = 0 \Leftrightarrow x = y$.
Hence, M_2 is satisfied.

M_3). Let $x, y \in X$ be arbitrary. Then, either $x = y \text{ or } x \neq y$

Now, if $x = y$, then, $d(x, y) = 0 = d(y, x)$

And if $x \neq y$, then, $d(x, y) = 1 = d(y, x)$.

Thus, in all cases, we have, $d(x, y) = d(y, x)$.

Hence, M_3 is satisfied.

M_4). Finally, let $x, y, z \in X$. Then, we have, either $= y \text{ or } x \neq y$.

Now, if $x = y$, then, $d(x, y) = 0 \text{ and } d(x, z) \geq 0, d(z, y) \geq 0$.

And so, $d(x, z) + d(z, y) \geq 0 = d(x, y)$ [$\because d(x, y) = 0$]

That is, $d(x, y) \leq d(x, z) + d(z, y) \text{ for } x = y$.

And if $x \neq y$, then, $d(x, y) = 1$. Further, in this case, z is different from at least one of x and y. So, it follows that at least one of $d(x, z)$ and $d(z, y)$ is equal to1. Hence, we have,

$$d(x, z) + d(z, y) \geq 1 = d(x, y) \ [\because d(x, y) = 1]$$

That is, $d(x,y) \leq d(x,z) + d(z,y)$ for $x \neq y$.

Thus, in all cases, we have, $d(x,y) \leq d(x,z) + d(z,y)$

Hence, M_4 is satisfied.

Thus all the metric conditions are satisfied by d and hence d is a metric for X and consequently (X, d) is a metric space.

Note: From the above example it is clear that every non-empty set can be made a metric space.

Example 3: Consider the set R of real numbers. Then, for a fixed $n \in N$, R^n **,** the set of all ordered n-tuples of real numbers, is the n-dimensional Euclidean space. Define $d: R^n \times R^n \to R$ for $x = (x_1, x_2, x_3, \dots, x_n)$, $y = (y_1, y_2, y_3, \dots, y_n) \in R^n$ such that

(i) $d(x,y) = (\sum_{i=1}^{n}(x_i - y_i)^2)^{\frac{1}{2}}$
(ii) $d(x,y) = \sum_{i=1}^{n}|x_i - y_i|$
(iii) $d(x,y) = \max\{|x_i - y_i|, i = 1, 2, 3, \dots, n\}$

Then d is a metric for $X = R^n$ in all the above cases.

Solution:

i) M_1] We have,

$$x = (x_1, x_2, x_3, \dots, x_n), y = (y_1, y_2, y_3, \dots, y_n) \in R^n$$

$$\Longrightarrow x_1, x_2, x_3, \dots, x_n, y_1, y_2, y_3, \dots, y_n \in R$$

$$\Longrightarrow x_i - y_i \in R, \forall i = 1, 2, 3, \dots, n$$

$$\Longrightarrow (x_i - y_i)^2 \geq 0, \forall i = 1, 2, 3, \dots, n$$

$$\Longrightarrow [\sum_{i=1}^{n}(x_i - y_i)^2]^{\frac{1}{2}} \geq 0$$

$$\Longrightarrow d(x,y) \geq 0$$

M_2] We have,

$$\begin{aligned} d(x,y) = 0 &\Leftrightarrow [\sum_{i=1}^{n}(x_i - y_i)^2]^{\frac{1}{2}} = 0 \\ &\Leftrightarrow (x_i - y_i)^2 = 0, \forall i = 1, 2, 3, \dots, n \\ &\Leftrightarrow x_i - y_i = 0, \forall i = 1, 2, 3, \dots, n \\ &\Leftrightarrow x_i = y_i, \forall i = 1, 2, 3, \dots, n \\ &\Leftrightarrow (x_1, x_2, x_3, \dots, x_n), = (y_1, y_2, y_3, \dots, y_n) \\ &\Leftrightarrow x = y \end{aligned}$$

Hence M_2 is satisfied.

M$_3$]. We have,

$$d(x,y) = \left[\sum_{i=1}^{n}(x_i - y_i)^2\right]^{\frac{1}{2}}$$

$$= \left[\sum_{i=1}^{n}(y_i - x_i)^2\right]^{\frac{1}{2}} = d(y,x), \forall x,y \in R^n$$

M$_4$]. For $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n), z = (z_1, z_2, \ldots, z_n) \in R^n$, we have,

$$d(x,y) = \left[\sum_{i=1}^{n}(x_i - y_i)^2\right]^{\frac{1}{2}} = \left[\sum_{i=1}^{n}(x_i - z_i + z_i - y_i)^2\right]^{\frac{1}{2}}$$

$$= \left[\sum_{i=1}^{n}(a_i + b_i)^2\right]^{\frac{1}{2}}, \quad a_i = x_i - z_i, b_i = z_i - y_i$$

$$\le \left[\sum_{i=1}^{n} {a_i}^2\right]^{\frac{1}{2}} + \left[\sum_{i=1}^{n} {b_i}^2\right]^{\frac{1}{2}}$$

$[\,By\ Minkowski's\ Inequality]$

$$= \left[\sum_{i=1}^{n}(x_i - z_i)^2\right]^{\frac{1}{2}} + \left[\sum_{i=1}^{n}(z_i - y_i)^2\right]^{\frac{1}{2}}$$

$$= d(x,z) + d(z,y)$$

Thus $d(x,y) \le d(x,z) + d(z,y), \forall\ x,y,z \in R^n$

So, d is a metric for $X = R^n$.

ii) Here, $d(x,y) = \sum_{i=1}^{n}|x_i - y_i|$, Now,

M$_1$]. We have,

$$x = (x_1, x_2, x_3, \ldots, x_n), y = (y_1, y_2, y_3, \ldots, y_n) \in R^n$$

$$\Longrightarrow x_1, x_2, x_3, \ldots, x_n, y_1, y_2, y_3, \ldots, y_n \in R$$

$$\Longrightarrow x_i - y_i \in R, \forall i = 1, 2, 3, \ldots, n$$

$$\Longrightarrow |x_i - y_i| \ge 0, \forall i = 1, 2, 3, \ldots, n$$

$$\Longrightarrow \sum_{i=1}^{n}|x_i - y_i| \ge 0$$

$$\Longrightarrow d(x,y) \ge 0$$

M$_2$]. We have,

$$d(x,y) = 0 \Leftrightarrow \sum_{i=1}^{n}|x_i - y_i| = 0$$

$$\Leftrightarrow |x_i - y_i| = 0, \forall i = 1, 2, 3, \ldots, n$$

$$\Leftrightarrow x_i - y_i = 0, \forall i = 1, 2, 3, \ldots, n$$

$$\Leftrightarrow x_i = y_i, \forall i = 1, 2, 3, \ldots, n$$

$$\Leftrightarrow (x_1, x_2, x_3, \ldots, x_n), = (y_1, y_2, y_3, \ldots, y_n)$$

$$\Leftrightarrow x = y$$

Hence M$_2$ is satisfied.

M$_3$]. We have,

$$d(x,y) = \sum_{i=1}^{n}|x_i - y_i| = \sum_{i=1}^{n}|y_i - x_i| = d(y,x), \forall x,y \in R^n$$

M$_4$]. For $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n), z = (z_1, z_2, \dots, z_n) \in R^n$, we have,

$$\begin{aligned} d(x,y) = \sum_{i=1}^{n}|x_i - y_i| &= \sum_{i=1}^{n}|x_i - z_i + z_i - y_i| \\ &= \sum_{i=1}^{n}|a_i + b_i|, \quad a_i = x_i - z_i, b_i = z_i - y_i \\ &\le \sum_{i=1}^{n}|a_i| + \sum_{i=1}^{n}|b_i| \\ &= \sum_{i=1}^{n}|x_i - z_i| + \sum_{i=1}^{n}|z_i - y_i| \\ &= d(x,z) + d(z,y) \end{aligned}$$

Thus $d(x,y) \le d(x,z) + d(z,y), \forall\, x,y,z \in R^n$

So, d is a metric for $X = R^n$.

iii) Here, $d(x,y) = \max\{|x_i - y_i|, i = 1,2,3,\dots,n\}$, Now,

M$_1$]. We have,

$$x = (x_1, x_2, x_3, \dots, x_n), y = (y_1, y_2, y_3, \dots, y_n) \in R^n$$

$$\Longrightarrow x_1, x_2, x_3, \dots, x_n, y_1, y_2, y_3, \dots, y_n \in R$$

$$\Longrightarrow x_i - y_i \in R, \forall i = 1,2,3,\dots,n$$

$$\Longrightarrow |x_i - y_i| \ge 0, \forall i = 1,2,3,\dots,n$$
$$\Longrightarrow \max\{|x_i - y_i|, i = 1,2,3,\dots,n\} \ge 0$$
$$\Longrightarrow d(x,y) \ge 0$$

M$_2$]. We have,

$$\begin{aligned} d(x,y) = 0 &\Leftrightarrow \max\{|x_i - y_i|, i = 1,2,3,\dots,n\} = 0 \\ &\Leftrightarrow |x_i - y_i| = 0, \forall i = 1,2,3,\dots,n \\ &\Leftrightarrow x_i - y_i = 0, \forall i = 1,2,3,\dots,n \\ &\Leftrightarrow x_i = y_i, \forall i = 1,2,3,\dots,n \\ &\Leftrightarrow (x_1, x_2, x_3, \dots, x_n), = (y_1, y_2, y_3, \dots, y_n) \\ &\Leftrightarrow x = y \end{aligned}$$

Hence M$_2$ is satisfied.

M$_3$]. We have,

$$\begin{aligned} d(x,y) &= \max\{|x_i - y_i|, i = 1,2,3,\dots,n\} \\ &= \max\{|y_i - x_i|, i = 1,2,3,\dots,n\} \\ &= d(y,x), \forall x,y \in R^n \end{aligned}$$

M$_4$]. For $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n), z = (z_1, z_2, \ldots, z_n) \in R^n$, we have,

$$\begin{aligned} d(x,y) &= \max\{|x_i - y_i|, i = 1,2,3,\ldots,n\} \\ &= \max\{|x_i - z_i + z_i - y_i|, i = 1,2,3,\ldots,n\} \\ &= \max\{|a_i + b_i|, i = 1,2,..,n\}, \quad a_i = x_i - z_i, b_i = z_i - y_i \\ &\le \max\{|a_i|, i = 1,2,..,n\} + \max\{|b_i|, i = 1,2,..,n\} \\ &= \max\{|x_i - z_i|, i = 1,2,3,\ldots,n\} + \max\{|z_i - y_i|, i = 1,2,3,\ldots,n\} \\ &= d(x,z) + d(z,y) \end{aligned}$$

Thus $d(x,y) \le d(x,z) + d(z,y), \forall\, x, y, z \in R^n$

So, d is a metric for $X = R^n$.

§ Some important Theorems

Theorem 1: ***If x, y, z are any three points in a metric space*** (X, d), then,

$$d(x,y) \ge |d(x,z) - d(z,y)|$$

Proof: By using M$_4$ and M$_3$, we have,

$$d(x,z) \le d(x,y) + d(y,z) = d(x,y) + d(z,y)$$

$$\Longrightarrow d(x,z) - d(z,y) \le d(x,y) \ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

Again, $d(z,y) \le d(z,x) + d(x,y) = d(x,z) + d(x,y)$

$$\Longrightarrow -\{d(x,z) - d(z,y)\} \le d(x,y) \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Now, (1) and (2) implies $d(x,y) \ge |d(x,z) - d(z,y)|$. □

Theorem 2: ***If x, x$_1$, y, y$_1$ are four points in a metric space*** (X, d), then,

$$|d(x,y) - d(x_1,y_1)| \le d(x,x_1) + d(y, y_1)$$

Proof: We have,

$$\begin{aligned} d(x,y) &\le d(x,x_1) + d(x_1,y) \\ &\le d(x,x_1) + d(x_1,y_1) + d(y_1,y) \end{aligned}$$

$$\Longrightarrow d(x,y) - d(x_1,y_1) \le d(x,x_1) + d(y,y_1) \ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

Again $\quad d(x_1,y_1) \le d(x_1,x) + d(x,y_1)$

$$\le d(x_1,x) + d(x,y) + d(y,y_1)$$

$$\Longrightarrow -[d(x,y) - d(x_1,y_1)] \le d(x,x_1) + d(y,y_1) \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Now (1) and (2) implies $|d(x,y) - d(x_1,y_1)| \leq d(x,x_1) + d(y,y_1)$.

§ Some important Definitions

1. ***Distance of a point from a set:***
   If x is a point and A is a subset of a metric space (X, d), then the distance of the point x from the set A is denoted by $d(x, A)$ and is defined by
   $$d(x,A) = \inf\{d(x,a) : a \in A\}$$
2. ***Distance between two sets***:
   If A and B are two subsets of a metric space (X, d), then the distance between the sets A and B is denoted by $d(A, B)$ and is defined by
   $$d(A,B) = \inf\{d(a,b) : a \in A,\ b \in B\}$$
3. ***Diameter of a set***:
   If A is a subset of a metric space (X, d), then the diameter of the set A denoted by $d(A)$ and is defined by
   $$d(A) = \sup\{d(a,b) : a, b \in A\}$$
4. ***Bounded metric space***:
   A metric space (X, d) is said to be bounded if there exists a real number $k > 0$ such that $d(A) \leq k$.

§ Example of a bounded metric space

Example: Let (X, d) be a metric space and $d^*: X \times X \to R$ such that $d^*(x,y) = \frac{Md(x,y)}{1+d(x,y)}$, $M > 0$. Then, d^* is a metric for X, called the induced metric for X induced by the metric d, and (X, d^*) is a bounded metric space with $d^*(X) \leq M$.

Solution: Since, (X, d) is a metric space, so, we have,

M_1) $d(x,y) \geq 0, \forall x, y \in X$

M_2) $d(x,y) = 0 \Leftrightarrow x = y$

M_3) $d(x,y) = d(y,x), \forall x, y \in X$

M_4) $d(x,y) \leq d(x,z) + d(z,y), \forall x, y, z \in X$

Now,

$$M_1^*)\ d(x,y) \geq 0, \forall x, y \in X, M > 0 \Longrightarrow 1 + d(x,y) \geq 1,\ Md(x,y) \geq 0$$

$$\Longrightarrow \frac{Md(x,y)}{1+d(x,y)} \geq 0, \forall x, y \in X$$

$$\Longrightarrow d^*(x,y) \geq 0, \forall x, y \in X$$

$$M_2^*)\ d^*(x,y) = 0 \Leftrightarrow \frac{Md(x,y)}{1+d(x,y)} = 0 \Leftrightarrow d(x,y) \geq 0 \Leftrightarrow x = y\ [by\ M_2]$$

$$M_3^*)\ d^*(x,y) = \frac{Md(x,y)}{1+d(x,y)} = \frac{Md(y,x)}{1+d(y,x)} = d^*(y,x), \forall x,y \in X \quad [By\ M_3]$$

$$M_4^*)\ d^*(x,y) = \frac{Md(x,y)}{1+d(x,y)} = M - \frac{M}{1+d(x,y)} = M - \frac{M}{1+d(x,z)+d(z,y)}$$

$$= \frac{Md(x,z)+Md(z,y)}{1+d(x,z)+d(z,y)}$$

$$= \frac{Md(x,z)}{1+d(x,z)+d(z,y)} + \frac{Md(z,y)}{1+d(x,z)+d(z,y)}$$

$$\leq \frac{Md(x,z)}{1+d(x,z)} + \frac{Md(z,y)}{1+d(z,y)}$$

$$= d^*(x,z) + d^*(z,y), \forall x,y,z \in X$$

Thus all metric conditions are satisfied by d^* and so it is a metric for X.

Further, $d^*(x,y) = \frac{Md(x,y)}{1+d(x,y)} = M - \frac{M}{1+d(x,y)} \leq M, \forall x,y \in X$. So, it follows that (X, d^*) is a bounded metric space with $d^*(x,y) \leq M, \forall x,y \in X$. □

Note: From the above example it is clear that every metric space, whether it is bounded or unbounded, is always a bounded metric space w. r. t. the induced metric.

§ ***Spheres or balls in metric spaces***: Open spheres and closed spheres

Let (X, d) be a metric space and $x_0 \in X$. Then, for $r > 0$,

i) The open sphere or open ball with centre $x_0 \in X$ and radius $r > 0$ is denoted by S(x_0, r) or B(x_0, r) or $S_r(x_0)$ or $B_r(x_0)$ and is defined by

$$S_r(x_0) = \{x \in X: d(x,x_0) < r\}$$

ii) The closed sphere or closed ball with centre $x_0 \in X$ and radius $r > 0$ is denoted by S$[x_0, r]$ or B$[x_0, r]$ or $S_r[x_0]$ or $B_r[x_0]$ and is defined by

$$S_r[x_0] = \{x \in X: d(x,x_0) \leq r\}$$

§ Describing Open and Closed spheres in a metric space

Example: *Describe open and closed spheres in the metric spaces R and R^2 for the usual metrics*.

Solution:

i) The usual metric d for R is defined as $(x, y) = |x - y|,\ \forall x, y \in R$.

Then the open sphere $S_r(x_0) = \{x \in R: d(x, x_0) < r\}$

$$= \{x \in R: |x - x_0| < r\}$$
$$= \{x \in R: x_0 - r < x < x_0 + r\}$$
$$= (x_0 - r, x_0 + r)$$

Further, the closed sphere $S_r[x_0] = \{x \in R: d(x, x_0) \le r\}$

$$= \{x \in R: |x - x_0| \le r\}$$
$$= \{x \in R: x_0 - r \le x \le x_0 + r\}$$
$$= [x_0 - r, x_0 + r]$$

That is, an open sphere in the usual metric space R is an open interval and a closed Sphere in the usual metric space R is a closed interval in R.

For example, $S_{1/2}(1) = \left\{x \in R: d(x, 1) < \frac{1}{2}\right\} = \left\{x \in R: |x - 1| < \frac{1}{2}\right\} = (\frac{1}{2}, \frac{3}{2})$

And, $S_1[-2] = \{x \in R: d(x, -2) \le 1\} = \{x \in R: |x + 2| \le 1\} = [-3, -1]$

ii) The usual metric d for R^2 is defined as

$$d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},\ \forall x = (x_1, y_1), y = (x_1, y_1) \in R^2 .$$

Then the open sphere centred at $x_0 = (h, k)$ and radius $r > 0$ is given by

$$S_r(x_0) = \{x = (x, y) \in R^2: d(x, x_0) < r\}$$
$$= \left\{x = (x, y) \in R^2: \sqrt{(x - h)^2 + (y - k)^2} < r\right\}$$
$$= \{x = (x, y) \in R^2: (x - h)^2 + (y - k)^2 < r^2\}$$
$$= the\ open\ disc\ with\ centre\ (h, k)\ and\ radius\ \ r > 0$$

Further, the closed sphere

$$S_r[x_0] = \{x = (x, y) \in R^2: d(x, x_0) \le r\}$$
$$= \left\{x = (x, y) \in R^2: \sqrt{(x - h)^2 + (y - k)^2} \le r\right\}$$
$$= \{x = (x, y) \in R^2: (x - h)^2 + (y - k)^2 \le r^2\}$$
$$= the\ closed\ disc\ with\ centre\ (h, k)\ and\ radius\ \ r > 0$$

Thus, an open sphere in the usual metric space R^2 is an open disc and a closed sphere in it is a closed disc in R^2.

§ Open sets in a metric space

In a metric space (X, d), a set $A \subseteq X$ is called an open set if for every $x_0 \in A$ there is an open sphere $S_r(x_0)$ such that $S_r(x_0) \subseteq A$. That is, if at every point in A, we can always construct an open sphere with some radius and centred at that point such that the sphere is contained in A, then A is an open set.

Example 1: *Every open interval on the real line R is an open set. But a closed or a semi-closed interval is not an open set.*

Solution: Consider the open interval (a, b) = I in R. We need to show that I is open.

Let $x \in \mathrm{I} = (\mathrm{a}, \mathrm{b})$ be arbitrary.

Now, $x \in \mathrm{I} = (\mathrm{a}, \mathrm{b}) \Longrightarrow a < x < b \Longrightarrow x - a, b - x > 0$. Let, $r = \min\{x - a, b - x\}$. Then, clearly, $S_r(x) = (x - r, x + r) \subseteq I$.

Thus, for every $x \in \mathrm{I} = (\mathrm{a}, \mathrm{b})$, there is an open sphere $S_r(x)$ such that $S_r(x) \subseteq I$. So, I is open.

On the other hand, if [a, b) = I, then, any open sphere $S_r(a) = (a - r, a + r)$ centred at the end-point a and for any arbitrary radius *r>0*, however small it may be, $S_r(a)$ contains infinitely many points which are not in I. For,$(a - r, a) \subseteq S_r(a) = (a - r, a + r)$, but $(a - r, a) \cap I = \emptyset$. That is, no point of the portion $(a - r, a)$ in $(a - r, a + r$ is in I. Hence, I, in this case is not open. Similarly, we can easily prove that a closed interval is also not an open set in the usual metric space R.

Example 2: *Every singleton set in a discrete metric space is open.*

Solution: Let (X, d) be a discrete metric space and $A = \{a\}$be a singleton set in X. We need to show that A is open.

Here, the only point in A is a. Now, by definition of discrete metric, for any $0 < r < 1$,

$S_r(a) = \{a\} = A \subseteq A$. So, by definition, A is open.

§ Theorems on Open sets

Theorem 1: *In a metric space (X, d), the null set Φ and X are open sets.*

Proof:

i) Φ is open:
Φ will be open if at every point in Φ we can construct an open sphere which is contained in Φ. But Φ being the null set, it contains no element at all. Hence the required criterion for Φ to be open is automatically satisfied. Hence Φ is open.

ii) X is open:
Let $x_0 \in X$ be arbitrary. Then for an arbitrary radius $r > 0$, the open sphere $S_r(x_0) = \{x \in X: d(x, x_0) < r\}$ is always, by its construction, contained in X.
So, by definition of open sets, X is an open set. #

Theorem 2: *In a metric space (X, d), every open sphere in X is an open set*.

Proof: Let $S_r(x_0) = G(say)$ be an open sphere in the metric space (X, d).

We need to prove that *G* is an open set. For this purpose, we need to prove that for every $x \in G$, there exists an open sphere $S_k(x)$ such that $S_k(x) \subseteq G$.

We have,

$$x \in G \Longrightarrow x \in S_r(x_0)$$

$$\Longrightarrow d(x, x_0) < r$$

$$\Longrightarrow k = r - d(x, x_0) > 0$$

We claim that the open sphere $S_k(x)$ centred at *x* is such that $S_k(x) \subseteq G$.

We have, $p \in S_k(x) \Longrightarrow d(x, p) < k$

$$\Longrightarrow d(x, p) < r - d(x, x_0)$$

$$\Longrightarrow d(x, p) + d(x, x_0) < r$$

$$\Longrightarrow d(p, x_0) < r \qquad [\because by\ M_4, d(p, x_0) < d(x, p) + d(x, x_0)]$$

$$\Longrightarrow p \in S_r(x_0) = G$$

So, it follows that $S_r(x_0) = G$ is an open set. Thus every open sphere in a metric space is open.

Theorem 3: *Arbitrary union of open sets in a metric space is again an open set*.

Proof: Let $\{G_\lambda : \lambda \in \Delta\}$ be an arbitrary collection of open sets in a metric space (X, d). Then, each $G_\lambda, \forall\ \lambda \in \Delta$, is open. We prove that $\bigcup_{\lambda\in\Delta} G_\lambda = G$ (say) is an open set.

We have,

$$x \in G = \bigcup_{\lambda\in\Delta} G_\lambda \Longrightarrow x \in G_\lambda\ for\ some\ \lambda \in \Delta$$

$$\Longrightarrow \exists\ S_k(x)\ such\ that\ S_k(x) \subseteq G_\lambda \quad [\because\ G_\lambda\ is\ open\ ]$$

$$\Longrightarrow \exists\ S_k(x)\ such\ that\ S_k(x) \subseteq G_\lambda \subseteq \bigcup_{\lambda\in\Delta} G_\lambda = G$$

$$\Longrightarrow \exists\ S_k(x)\ such\ that\ S_k(x) \subseteq G$$

So, by definition of open sets, it follows that $\bigcup_{\lambda\in\Delta} G_\lambda = G$ is open.

Thus the union of an arbitrary collection of open sets in a metric space is also an open set. #

Theorem 4: *Finite intersection of open sets in a metric space is an open set*.

Proof: Let $\{G_i: i = 1,2,3, \ldots, n\}$ be a finite collection of open sets in a metric space (X, d). Then, for each $i = 1,2,3, \ldots, n$, G_i is open. We prove that $\bigcap_{i=1}^{n} G_i = G$ (say) is an open set.

We have,

$$x \in G = \bigcap_{i=1}^{n} G_i \implies G_i \text{ is open }, \forall\, i = 1,2,3, \ldots, n$$

$$\implies \exists\, S_{r_i}(x) \text{ s.t. } S_{r_i}(x) \subseteq G_i , \forall\, i = 1,2,3, \ldots, n \quad [\because\ G_i \text{ is open}]$$

Thus we get a finite class $\{S_{r_i}(x) :\ S_{r_i}(x) \subseteq G_i , \forall\, i = 1,2,3, \ldots, n \}$ of concentric spheres of the metric space (X, d) centred at the point *x*. So, if $r = \min\{r_1, r_2, \ldots, r_n\}$, then, clearly, the open sphere $S_r(x)$ is such that

$$S_r(x) \subseteq S_{r_i}(x) \subseteq G_i , \forall\, i = 1,2,3, \ldots, n .$$

Now, $S_r(x) \subseteq G_i , \forall\, i = 1,2,3, \ldots, n \implies S_r(x) \subseteq \bigcap_{i=1}^{n} G_i = G$.

Thus, for each $x \in G$, $\exists S_r(x)$ such that $S_r(x) \subseteq G$.

So, by definition of open sets, it follows that $\bigcap_{i=1}^{n} G_i = G$ is open.

Thus the intersection of a finite collection of open sets in a metric space is also an open set. ♯

Note: Infinite intersection of open sets in a metric space is not necessarily open. E.g., consider the infinite collection $\left\{I_n: I_n = \left(-\frac{1}{n}, \frac{1}{n}\right), n \in \mathrm{N}\right\}$ of open intervals in the usual metric space R. Since, every open interval on the real line is open, so, it is an infinite collection of open sets in R. Here, $\bigcap_{i=1}^{\infty} I_n = \{0\}$ which is not an open set in R. For, any open sphere $S_r(0) = (-r, r)$ centred at 0 is such that it contains infinite number of real numbers which are not members of $\{0\}$.

Theorem 5: *A set G in a metric space (X, d) is open if and only if G is a union of open spheres in X*.

Proof: Let G be an open set in (X, d).

Then, if G is empty, then G is the union of an empty collection of open spheres in X.

If G is non-empty, then, G being open, for each $x \in G$, $\exists$ an open sphere $S_r(x)$ such that $S_r(x) \subseteq G$. Then for such open spheres at every point in G, we have that $\bigcup_{x \in G} S_r(x) = G$. Therefore, G is a union of open spheres.

Conversely, let, G is a union of open spheres in X. Then, since, every open sphere is an open set, so G is a union of open sets. Again, since arbitrary union of open sets is open, so it follows that G is open.

§ *Limit points or accumulation points:* derived sets

Let (X, d) be a metric space and $A \subseteq X$. Then a point $x \in X$ is called a *limit* or *accumulation or cluster point* of the set A if for every open sphere $S_r(x)$, $S_r(x)$ contains a point of A other than, possibly, x.

The set of all the limit points of A is called the derived set of A and it is generally denoted by D(A). That is, D(A) = { $x \in X$: x is a limit point of A}.

Example 1: Consider the set A= {1, 1/2, 1/3, 1/4, ………} in the real line R with the usual metric.