MATHEMATICAL LOGIC

Introduction

Logic plays a very important role in the study of mathematics. Mainly the study of modern mathematics is quite impossible without the help of logic. Precisely speaking, the logical laws are very handy and powerful tools for mathematics. Mathematics is highly dependent on logic so far as mathematical implications, equivalences and the art of decision making are concerned. Application of logical rules has given a good momentum to the subject. It is not a recent trend in mathematics; history of mathematical developments shows that logic got tremendous importance in mathematics from the very beginning of the subject. Many mathematicians, scientists and great thinkers of human civilizations honestly confessed it in various moments and termed mathematics as a subject equivalent to logic. In this context we can recall here the most important statements made by the great physicist Albert Einstein and the great philosopher Bertrand Russell. Albert Einstein said, "***Pure mathematics is the poetry of logic***" and Bertrand Russell exclaimed, "***Mathematics is a subject identical with logic***". These two statements are quite sufficient to understand the indispensability of the role of logic to the study of mathematics.

Mathematical logic has two aspects. Those are (1) application of logical reasoning to mathematics and (2) study of logic with the help of mathematics. Both the aspects are important in mathematical viewpoint as both contribute a great deal in the development of mathematics.

§ *Statements*

Statements are sentences which can be judged to be either true or false but not both and neither. The statements are actually assertive or declarative sentences whose verbal meaning can be asserted or declared as TRUE or FALSE in exclusive sense.

Statements are generally denoted by p, q, r etc. or by A, B, C, … etc. The truth or falsity of a statement is called its *truth value*. When the truth value of a statement is true, then we simply denote it by the letter T and F otherwise.

Some examples of statements with their truth values

Example 1

(i) The Earth moves round the Sun.(T)
(ii) 1 is a prime number (F).
(iii) 7 > 4 (T)
(iv) New Delhi is the capital of India (T)
(v) The Moon is a satellite of the Mars (F)

Some examples which are non-statements

Example 2

(i) Where are you going?
(ii) Go there and bring the chair.
(iii) X > 0.
(iv) What a great surprise!
(v) Hurrah! We have won the match.

§ *Compound statements: Sentential or binary Connectives*

Compound statements are statements obtained by joining two or more statements with connectives. The connectives used in this purpose are called logical or sentential connectives. The commonly used sentential connectives are "AND", "OR", "If…, then ....", and "If and only if". Symbolically these sentential connectives are denoted by $\wedge, \vee, \rightarrow$ *and* $\leftrightarrow$ respectively. These sentential connectives $\wedge, \vee, \rightarrow$ *and* $\leftrightarrow$ are respectively termed as the *conjunction*, *disjunction*, *conditional* and *bi-conditional*. The connectives $\rightarrow$ and $\leftrightarrow$ are also denoted by $\Longrightarrow$ and $\Leftrightarrow$ respectively. As these connectives connect two statements, so these are also known as the binary connectives.

A statement having no sentential connective is called a *prime* or *atomic* or a *simple statement*. Clearly, a prime statement is a simple declarative sentence. So, it follows that a compound statement contains more than one prime statement.

On the other hand, as the compound statements are statements themselves, so they also have some truth-values depending on the truth-values of its prime or simple statements. For different allocations of truth-values to the prime statements in a compound statement, generally, we get different truth-values of the compound statement. The truth-values of the compound statement for different allocations of truth-values to its prime statements can be clearly and elaborately demonstrated in a single table. This table is known as a *truth-table*. That is, a truth-table is a table that clearly depicts the truth-values of a compound statement for different allocations of truth-values to its prime statements.

We now separately study the above mentioned four types of compound statements along with their truth-tables one by one.

(1) *Conjunction of two statements*: If two statements p and q are connected by the sentential connective "AND ($\wedge$)", then the compound statement "p *and* q" or "$p \wedge q$" thus obtained is called the conjunction of the two statements p and q.
The truth-table for the conjunction "$p \wedge q$" is given below:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Note: The truth-value of the conjunction of two statements is T when the truth-values of both the component statements are also T and otherwise its truth-value is F.

(2) *Disjunction of two statements*: If two statements p and q are connected by the sentential connective "OR (⋁)", then the compound statement "*p or q*" or "$p \lor q$" thus obtained is called the conjunction of the two statements p and q.
The truth-table for the conjunction "$p \lor q$" is given below:

p	q	$p \lor q$
T	T	T
T	F	T
F	T	T
F	F	F

Note: The truth-value of the disjunction of two statements is F when the truth-values of both the component statements are also F and otherwise its truth-value is always T.

(3) *Conditional statement of two statements*: If two statements p and q are connected by the sentential connective "If…, then… (→)", then the compound statement "If *p, then q*" or "$p \rightarrow q$" thus obtained is called the conditional statement of the two statements p and q. Here p is called the *antecedent* and q is called the *consequence* of the conditional statement "$p \rightarrow q$".
The truth-table for the conditional statement "$p \rightarrow q$" is given below:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Note: The truth-value of a conditional statement is F when the truth-values of the *antecedent* and the *consequence* are respectively T and F and otherwise its truth-value is always T.

(4) *Bi-conditional statement of two statements*: If two statements p and q are connected by the sentential connective "If and only if (↔)" or "iff", then the compound statement " *p iff q*" or "$p \leftrightarrow q$" thus obtained is called the bi-conditional statement of the two statements p and q.
The truth-table for the bi-conditional statement "$p \leftrightarrow q$" is given below:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Note: The truth-value of a bi-conditional statement is T when the truth-values of both the prime statements are same and otherwise its truth-value is F.

§ *Negation of a statement*

If p is a statement, then the statement "It is not true that p" or "It is false that p" is called the negation of the statement p and it is denoted by "$\sim p$".

One must note that the negation of a statement is not the opposite of the statement. If p is the statement: "The Sun rises in the East", then its negation is not the statement: "The Sun rises in the West", but the statement: 'It is not true that the Sun rises in the East' or "The Sun not rises in the East".

The truth-value of the negation of a statement is opposite to that of the original statement. It can be demonstrated by the following table:

p	$\sim p$
T	F
F	T

§ *Tautologies*

A compound statement is called a *tautology* if its truth-value is always T for all possible truth-value allocations to its prime statements. That is, whatever truth-values have been allocated to the prime statements of a compound statement, the truth-value of the compound statement is always T for a tautology. A tautology is also called a *valid statement*.

Tautologies are very much useful in establishing the validities of mathematical statements.

Below we give a list of tautologies, though not exhaustive, which are widely used in mathematics.

A. Tautological bi-conditionals

1. The idempotent laws: $(p \land p) \leftrightarrow p$, $(p \lor p) \leftrightarrow p$
2. Commutative laws: $(p \land q) \leftrightarrow (q \land p)$, $(p \lor q) \leftrightarrow (q \lor p)$
3. De-Morgan's laws: $\sim (p \land q) \leftrightarrow (\sim p \lor \sim q)$, $\sim(p \lor q) \leftrightarrow (\sim p \land \sim q)$
4. Law of double negation: $\sim (\sim p) \leftrightarrow p$
5. Law of contrapositive: $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
6. Associative laws: $(p \land q) \land r \leftrightarrow p \land (q \land r)$, $(p \lor q) \lor r \leftrightarrow p \lor (q \lor r)$
7. Distributive laws:

   $p \land (q \lor r) \leftrightarrow (p \land q) \lor (p \land r)$, $p \lor (q \land r) \leftrightarrow (p \lor q) \land (p \lor r)$

   $p \land (q \land r) \leftrightarrow (p \land q) \land (p \land r)$, $p \lor (q \lor r) \leftrightarrow (p \lor q) \lor (p \lor r)$
8. Law of equivalence for implication and disjunction:

   $$(p \rightarrow q) \leftrightarrow (\sim p \lor q)$$
9. Law of negation for implication:

   $$\sim (p \rightarrow q) \leftrightarrow (p \land \sim q)$$

B. Tautological Conditionals:

1. Laws of simplification: $(p\wedge q) \to p$, $(p\wedge q) \to q$
2. Laws of addition: $p\to (p\vee q)$, $q\to (p\vee q)$
3. Law of Detachment: $p\wedge(p \to q) \to q$
4. Modus tollendo tollens: $\sim q\wedge(p \to q) \to \sim p$
5. Modus tollendo ponens: $\sim p\wedge(p\vee q) \to q$
6. Law of Adjunction: $(p\wedge q) \to (p\vee q)$
7. Law of Hypothetical Syllogism: $(p\to q) \wedge (q \to r) \to (p \to r)$
8. Law of Exportation: $[\, p \wedge q \to r] \to [p \to (q \to r)]$
9. Law of Importation: $[p \to (q \to r)] \to [\, p \wedge q \to r]$

C. Two further Tautologies:

1. Law of Excluded middle: $p\vee\sim p$
2. Law of contradiction: $\sim(p\wedge\sim p)$

§ Construction of truth-tables for some tautologies

Below we construct some truth-tables for a few number of tautologies mentioned above for our confirmation.

1. Truth-table for the tautology $(p\wedge p) \leftrightarrow p$ [Idempotent Law]

p	p∧p	(p∧p) ↔ p
T	T	T
F	F	T
		*

OR

(p	∧	p)	↔	p
T	T	T	T	T
F	F	F	T	F
			*	

2. Truth-table for the tautology ~ (p⋀q) ↔ (~p⋁~q) [De-Morgan’s Law]

p	q	~p	~q	p⋀q	~ (p⋀q)	~p⋁~q	~ (p⋀q) ↔ (~p⋁~q)
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T
							*

3. Truth-table for the tautology (p ⋀q)⋀ r ↔ p⋀(q⋀r) [Associative Law]

p	q	r	p ⋀q	q⋀r	(p ⋀q)⋀ r	p⋀(q⋀r)	(p ⋀q)⋀ r ↔ p⋀(q⋀r)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	F	F	T
T	F	F	F	F	F	F	T
F	T	T	F	T	F	F	T
F	T	F	F	F	F	F	T
F	F	T	F	F	F	F	T
F	F	F	F	F	F	F	T
							*

OR

(p	⋀	q)	⋀	r	↔	p	⋀	(q	⋀	r)
T	T	T	T	T	T	T	T	T	T	T
T	T	T	F	F	T	T	F	T	F	F
T	F	F	F	T	T	T	F	F	F	T
T	F	F	F	F	T	T	F	F	F	F
F	F	T	F	T	T	F	F	T	T	T
F	F	T	F	F	T	F	F	T	F	F
F	F	F	F	T	T	F	F	F	F	T
F	F	F	F	F	T	F	F	F	F	F
					*					

4. Truth-table for the tautology p ⋁(q⋀ r) ↔ (p⋁q)⋀(p⋁r) [Law of distribution]

p	⋁	(q	⋀	r)	↔	(p	⋁	q)	⋀	(p	⋁	r)
T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	T	F	F	T	T	T	T	T	T	T	F
T	T	F	F	T	T	T	T	F	T	T	T	T
T	T	F	F	F	T	T	T	F	T	T	T	F
F	T	T	T	T	T	F	T	T	T	F	T	T
F	F	T	F	F	T	F	T	T	F	F	F	F
F	F	F	F	T	T	F	F	F	F	F	T	T
F	F	F	F	F	T	F	F	F	F	F	F	F
					*							

5. Truth-table for the tautology (p∧q) → p [Law of simplification]

(p	∧	q)	→	p
T	T	T	T	T
T	F	F	T	T
F	F	T	T	F
F	F	F	T	F
			*	

6. Truth-table for the tautology (p→ q) ∧ (q → r) → (p → r) [Law of Hypothetical Syllogism]

(p	→	q)	∧	(q	→	r)	→	(p	→	r)
T	T	T	T	T	T	T	T	T	T	T
T	T	T	F	T	F	F	T	T	F	F
T	F	F	F	F	T	T	T	T	T	T
T	F	F	F	F	T	F	T	T	F	F
F	T	T	T	T	T	T	T	F	T	T
F	T	T	F	T	F	F	T	F	T	F
F	T	F	T	F	T	T	T	F	T	T
F	T	F	T	F	T	F	T	F	T	F
							*			

7. Truth-table for the tautology [p ∧ q → r] → [p → (q → r)] (Law of Exportation)

[p	∧	q	→	r]	→	[p	→	(q	→	r)]
T	T	T	T	T	T	T	T	T	T	T
T	T	T	F	F	T	T	F	T	F	F
T	F	F	T	T	T	T	T	F	T	T
T	F	F	T	F	T	T	T	F	T	F
F	F	T	T	T	T	F	T	T	T	T
F	F	T	T	F	T	F	T	T	F	F
F	F	F	T	T	T	F	T	F	T	T
F	F	F	T	F	T	F	T	F	T	F
					*					

Note: In the above truth-tables, we observe that the final columns * contain only the entry T. These ensure us that the truth-value of the compound statement for any combination of truth-values of their prime statements is always True. The truth-tables ensure us that the compound statements are really tautologies.

§ Contradictions:

A compound statement is called a *contradiction* if its truth-value is always F for all possible truth-value allocations to its prime statements. For example, ~p∧p is a contradiction.

§ Quantifiers

Consider the following statements:

1. Every rational number is a real number
2. Some real numbers are integers
3. Every human being is mortal

The above statements may also be respectively stated as-

1. For every x, if x is a rational number, then x is a real number.
2. For some x, x is a real number and x is an integer.
3. For each x, if x is a human being, then x is mortal.

These statements are translations of the above statements and are more formal in logical viewpoint as these can easily be symbolized to logical statement forms.

The universal quantifier

The phrase “For every x” is called the *universal quantifier*. The phrases “For each x” or “For all x” also carry the same meaning as that of the phrase “For every x”. The universal quantifier quantifies that a property P(x) involving x holds for each and every element of a set, called the set of context, mentioned in the statement.