Unit-1: Topic-2: Functions or mappings

2.0: Introduction

The notion of functions is not only essential, but also very much fundamental in the study of mathematics. A function in mathematics can be compared to a machine in an industry which processes raw materials fed into it and manufactures our desired final finish products. In a very similar way, a function in mathematics processes the elements in a set and finally gives us some unique elements of our requirement. In a very crude sense, a function is a rule which normally connects, corresponds, associates or relates (in mathematical sense) the elements in a set with some elements in another set in a very unique way.

We have learnt many things like limits, continuity, differentiability, integrability etc. about real functions in calculus. Here we mainly discuss about the general theory of functions in algebraic viewpoint which are applicable to all types of functions, i.e. not restricted to real or real-valued functions only, found in all branches of mathematics.

2.1: Some basic definitions, notations and examples:

Definition* 2.1.01**: Let A and B two non-empty sets. Then a correspondence/association rule or a relation f which corresponds/associates/relates each element of the set A to a unique element in the set B is called a ***function or a ***mapping*** from the set A to the set B. More specifically, a function is a relation that relates every element in a set to a unique element in another set.

Notations: If f is a ***mapping*** from A to B, then we denote this fact by $f:\mathrm{A}\to\mathrm{B}$ or by $\mathrm{A}\xrightarrow{f}\mathrm{B}$. Further if f relates the element $a\in\mathrm{A}$ to the element $b\in\mathrm{B}$, then, b is called the f -***image*** or simply ***the image of*** a and conversely, a is called ***a pre-image of*** b. In this case we write $f(a)=b$. An element in B may have no, one or more than one pre-images in A. The pre-image(s) of $b\in\mathrm{B}$ is/are denoted by $f^{-1}(b)$ and is defined by

$$f^{-1}(b)=\{x\in\mathrm{A}:\ f(x)=b\}$$

Definition 2.1.02: If $f:\mathrm{A}\to\mathrm{B}$, then A and B are respectively called the ***domain*** and the ***co-domain*** of the function f **.** Also the set

$f(\mathrm{A})=\{b\in\mathrm{B}:b=f(a)\ \ for\ \ some\ a\in\mathrm{A}\}$ is called the ***range of*** f .

***Definition 2.1.03*: Equality of mappings**: Two mappings $f:\mathrm{A}\to\mathrm{B}$ and $g:\mathrm{A}\to\mathrm{B}$ are said to be equal, denoted as $f=g$, if $f(x)=g(x),\ \forall x\in A$. That is, to prove that $f=g$ we need to show that:

(i) Domain of f = Domain of g

(ii) Co-domain of f = co-domain of g

And (iii) $f(x) = g(x), \forall x \in A$

Example 2.1.04: The following functions f and g are equal:

(i) $f(x) = 1, \; g(x) = \sin^2 x + \cos^2 x$

(ii) $f(x) = |x^2|, \; g(x) = x^2$

But the functions $f(x) = 1$ and $g(x) = \sec^2 x - \tan^2 x$ are not equal.

For, $Domain\ of\ \ f = R \neq Domain\ of\ \ g = R - \left\{ \frac{n\pi}{2} : n \in \mathrm{Z},\ n\ \ is\ \ odd \right\}$

2.2: Different types of mappings or functions:

A mapping $f : \mathrm{A} \to \mathrm{B}$ is said to be

(i) A ***one-one*** or an ***injective mapping*** or an ***injection*** if different elements in A have different *f*-images in B.

Mathematically, f is one-one if

$$x, y \in \mathrm{A},\ x \neq y \Rightarrow f(x) \neq f(y)$$

Or equivalently, if $x, y \in \mathrm{A},\ f(x) = f(y) \Rightarrow x = y$

(ii) A ***many-one or many-to-one mapping*** if two or more than two elements in the domain A have the same *f*-image in B.

(iii) An ***onto*** or a ***surjective mapping*** or a ***surjection*** if $f(\mathrm{A}) = \mathrm{B}$, i.e. every element in B has at least one pre-image in A. More precisely, f is an onto mapping if for every $y \in \mathrm{B}$, there is $x \in \mathrm{A}$ such that $f(x) = y$.

(iv) An ***into mapping*** if $f(\mathrm{A}) \neq \mathrm{B}$, i.e. there is at least one element in B which has no any pre-image in A under *f*.

(v) A ***bijection*** or a ***bijective mapping*** or a ***one-to-one correspondence*** if f is both an injection and a surjection i.e., both one-one and onto.

Example 2.2.01. The mapping $f : R \to R$ defined by $f(x) = ax + b,\ a(\neq 0), b \in R$ is a bijection.

For, (i) We have,

$$\begin{aligned} f(x) = f(y) &\Rightarrow ax + b = ay + b \\ &\Rightarrow a.(x - y) = 0 \\ &\Rightarrow x - y = 0 \qquad [\because a \neq 0] \\ &\Rightarrow x = y \end{aligned}$$

Therefore, *f* is an injection or a one-one mapping.

(ii) Let y be an arbitrary element in the co-domain R of f. If possible, let, there be an element x in the domain R such that $y = f(x) = ax + b$. Then,

$$\begin{aligned} y = f(x) = ax + b &\Rightarrow ax = y - b \\ &\Rightarrow x = \frac{y-b}{a} \in R \left[\begin{aligned} &\because y, b, a(\neq 0) \in R \Rightarrow y - b, a(\neq 0) \in R \\ &\Rightarrow \frac{y-b}{a} \in R \end{aligned} \right] \\ &\Rightarrow the\ pre-image\ of\ y \in R(co-domain) is\ x \in R(domain) \end{aligned}$$

So, f is a surjection or is an onto mapping.

Example 2.2.02 $f : \mathrm{N} \to \mathrm{N}\ \ s.t.\ \ f(x) = 2x^2 + 1$ is one-one but not onto. Further the mapping g given by $g : \mathrm{Z} \to \mathrm{N}\ \ s.t.\ \ g(x) = 2x^2 + 1$ is neither one-one nor onto.

Proof: (i) $f : \mathrm{N} \to \mathrm{N}\ \ s.t.\ \ f(x) = 2x^2 + 1$ is one-one:

We have,

$$\begin{aligned} f(x) = f(y), x, y \in \mathrm{N} &\Rightarrow 2x^2 + 1 = 2y^2 + 1 \\ &\Rightarrow 2(x^2 - y^2) = 0 \\ &\Rightarrow (x + y).(x - y) = 0 \\ &\Rightarrow x - y = 0 \qquad [\because x, y \in \mathrm{N} \Rightarrow x + y > 0] \\ &\Rightarrow x = y \end{aligned}$$

So, it follows that $f : \mathrm{N} \to \mathrm{N}\ \ s.t.\ \ f(x) = 2x^2 + 1$ is a one-one mapping.

Further,

$$y = f(x) = 2x^2 + 1 \Rightarrow x^2 = \frac{y-1}{2}$$

$$\Rightarrow x = \sqrt{\frac{y-1}{2}} \quad \notin \mathrm{N} \ \ for \ \ y = 1, 2, 4, 6, ...$$

$$\Rightarrow y = 1, 2, 4, 6, \ ...... \in \mathrm{N} \, have \ no \ \ pre - image \ in \ domain$$

So, f is not an onto mapping.

(ii) $g : \mathrm{Z} \to \mathrm{N} \ \ s.t. \ \ g(x) = 2x^2 + 1$ is neither one-one nor onto:

Here, $\pm 1, \pm 2, \pm 3, \pm 4, ... \in \mathrm{Z} \ and \ 1 \neq -1, \ 2 \neq -2, \ 3 \neq -3, \ 4 \neq -4$

But $f(1) = f(-1), f(2) = f(-2), f(3) = f(-3), f(4) = f(-4), ...$

Therefore, $g : \mathrm{Z} \to \mathrm{N} \ \ s.t. \ \ g(x) = 2x^2 + 1$ is not one-one.

Again, as in (i) we can easily establish that $g(x) = 2x^2 + 1$ is not onto.

2.3: *Definition of functions as a set of ordered pairs*:

A function $f : \mathrm{A} \to \mathrm{B}$ is a subset of $\mathrm{A} \times \mathrm{B}$ such that

(i) $a \in \mathrm{A} \Rightarrow \exists \ (a, b) \in f \ \ for \ some \ b \in \mathrm{B}$

(ii) $(a, b), \ (a, c) \in f \ \Rightarrow b = c$

Condition (i) implies that every element in A is related to some element in B while (ii) implies that an element in A is related to a unique element in B.

Example 2.3.01: If $\mathrm{A} = \{a, b, c\}, \ \ \mathrm{B} = \{x, y, z, w\}$, then following are some functions from A to B.

(i) $f_1 = \{(a, w), (b, y), (c, x)\}$

(ii) $f_2 = \{(a, x), (b, x), (c, x)\}$

(iii) $f_3 = \{(a, x), (b, y), (c, z)\}$

But the following are not functions from A to B:

(a) $f_4 = \{(a, w), (b, y), (c, x), (a, z)\}$.Here $a \in \mathrm{A}$ is related to two different $w, z \in \mathrm{B}$.

(b) $f_5 = \{(a, w), (b, y)\}$. Here $c \in \mathrm{A}$ is not related to any element in B.

2.4: Some particular types of functions:

Definition 2.4.01: ***Transformations or operators*:**

A function from a set A to itself is called a ***transformation or an operator*** on the set A. i.e., if the domain and codomain of a function are same, then it is called a transformation.

Definition 2.4.02: ***Identity functions*:**

The mapping $\mathrm{I}:\mathrm{A}\to\mathrm{A}$ defined by $\mathrm{I}(x)=x,\ \forall\, x\in\mathrm{A}$, is called the ***identity function*** on A. The identity mapping on A is generally denoted by I_{A}.

Definition 2.4.03: ***Constant functions*:**

A mapping $f:\mathrm{A}\to\mathrm{B}$ defined by $f(x)=k,\ \forall\, x\in\mathrm{A}$, where $k\in\mathrm{B}$ is fixed, is called a ***constant function*** from A to B.

2.5: ***Product or composition of mappings*:**

If $f:\mathrm{A}\to\mathrm{B}$ and $g:\mathrm{B}\to C$, then their product or composition is denoted by $g\circ f$ and it is defined to be a function from A to C such that

$$(g\circ f)(x)=g\big(f(x)\big),\quad \forall\, x\in\mathrm{A}$$

We must note here that $g\circ f$ is defined when and only when

Co-domain of f = Domain of g

Example 2.5.01: Let $f:R\to R$ and $g:R\to R$, where R is the set of real numbers, such that

$$f(x)=2x+3 \text{ and } g(x)=x^2-5,\ \forall x\in R$$

Then, $(g\circ f)(x)=g(f(x))=g(2x+3)=(2x+3)^2-5=4x^2-12x+4$

And $(f\circ g)(x)=f(g(x))=f(x^2-5)=2(x^2-5)+3=2x^2-7$

Hence, $g\circ f\neq f\circ g$

Remark: The above example also shows that composition of mappings is not necessarily commutative.

Theorem: 2.5.02: *The composition of functions or mappings is always associative. i.e.,* $h\circ(g\circ f)=(h\circ g)\circ f$

Proof: We first recall that two mappings are equal when (i) their domains and codomains are same and (ii) the functional values of both the mappings at every point of the domain are equal. So, keeping in mind the above criteria we prove the theorem.

Let $f : \mathrm{A} \to \mathrm{B}$, $g : \mathrm{B} \to C$ and $h : C \to D$.

Then,

$$
\begin{aligned}
& f : \mathrm{A} \to \mathrm{B},\ g : \mathrm{B} \to C,\ h : C \to D \\
& \qquad \Rightarrow g \circ f : \mathrm{A} \to C,\ h \circ g : \mathrm{B} \to D \\
& \qquad \Rightarrow h \circ (g \circ f) : \mathrm{A} \to D,\ (h \circ g) \circ f : \mathrm{A} \to D \\
& \qquad \Rightarrow \textit{Domains and codomains of } h \circ (g \circ f) \textit{ and } (h \circ g) \circ f \textit{ are same}
\end{aligned}
$$

Further, we have,

$$
\begin{aligned}
[h \circ (g \circ f)](x) &= h[(g \circ f)(x)] \quad [\textit{ by definition of composite mappings}] \\
&= h[g(f(x))] \quad [\textit{ by definition of composite mappings}] \\
&= h[g(y)] \quad \textit{where } y = f(x) \\
&= (h \circ g)(y) \\
&= (h \circ g)(f(x))] \quad \textit{putting } y = f(x) \\
&= [(h \circ g) \circ f](x), \quad \forall x \in \mathrm{A} \\
\therefore\ h \circ (g \circ f) &= (h \circ g) \circ f
\end{aligned}
$$

#

Theorem: 2.5.03: *If* $f : \mathrm{A} \to \mathrm{B}$ *and* $g : \mathrm{B} \to C$ *, then*

(i) *If f and g are both one-to-one, then* $g \circ f$ *is also one-to-one*

(ii) *If f and g are both onto, then so is* $g \circ f$

Proof:

(i) We have,

$$
\begin{aligned}
& (g \circ f)(x) = (g \circ f)(y),\ x, y \in \mathrm{A} \\
& \Rightarrow g(f(x)) = g(f(y)) \quad [\textit{by definition of composition of mappings}] \\
& \Rightarrow f(x) = f(y)) \quad [\because g \textit{ is one-to-one}] \\
& \Rightarrow \quad x = y \quad [\because f \textit{ is one-to-one}]
\end{aligned}
$$

Therefore, $g \circ f$ is one-to-one.

(ii)Let $c \in C$, the co-domain of $g \circ f$, be arbitrary.

Now, $g : \mathrm{B} \to C$ *is onto,* $c \in C \Rightarrow \exists\ b \in \mathrm{B}$ *s.t.* $g(b) = c$

Further,

$$f : \mathrm{A} \to \mathrm{B} \textit{ is onto}, b \in \mathrm{B} \Rightarrow \exists\ a \in \mathrm{A}\, s.t.\ f(a) = b$$
$$\Rightarrow \exists\ a \in \mathrm{A}\, s.t.\ g(f(a)) = g(b) = c$$
$$\Rightarrow \exists\ a \in \mathrm{A}\, s.t.\ (g \circ f)(a) = c$$

So, it follows that $g \circ f$ is onto.

Theorem: 2.5.04: *If* $f : \mathrm{A} \to \mathrm{B}$, *then* $f \circ \mathrm{I_A} = f$ *and* $\mathrm{I_B} \circ f = f$ *where* $\mathrm{I_A}$ *and* $\mathrm{I_B}$ *are identity mappings on A and B respectively*.

Proof: Here,

$$f : \mathrm{A} \to \mathrm{B}\ , \mathrm{I_A} : \mathrm{A} \to \mathrm{A}, \mathrm{I_B} : \mathrm{B} \to \mathrm{B} \Rightarrow f \circ \mathrm{I_A} : \mathrm{A} \to \mathrm{B}, \mathrm{I_B} \circ f : \mathrm{A} \to \mathrm{B}$$

Thus the *domain and codomain of* $f \circ \mathrm{I_A}$ *and* f *are same*.

And the *domain and codomain of* $\mathrm{I_B} \circ f$ *and* f *are same*.

Further, $(f \circ \mathrm{I_A})(x) = f(\mathrm{I_A}(x)) = f(x), \forall x \in \mathrm{A}$

Therefore, $f \circ \mathrm{I_A} = f$

In a very similar way we can prove that $\mathrm{I_B} \circ f = f$

2.6: Inverse mappings:

Inverse mappings are directly obtained from bijective mappings or one-to-one correspondences (one-one and onto mappings). Those are also defined in terms of bijective mappings only.

Theorem: 2.6.01: ***If*** $f : \mathrm{A} \to \mathrm{B}$ ***is one-one onto, then*** $f^{-1} : \mathrm{B} \to \mathrm{A}$ ***is also one-one onto. i.e., the inverse of a bijective mapping is also bijective.***

Proof: Let $f : \mathrm{A} \to \mathrm{B}$ is a bijective mapping and $f^{-1} : \mathrm{B} \to \mathrm{A}$ be its inverse such that

$$f^{-1}(b) = a \text{ if and only if } f(a) = b$$

We now need to prove that $f^{-1} : \mathrm{B} \to \mathrm{A}$ is also bijective i.e. one-one and onto.

We have,

$$f^{-1}(x) = f^{-1}(y),\ x, y \in \mathrm{B} \Rightarrow f^{-1}(x) = f^{-1}(y) = a,\ \textit{for some } a \in \mathrm{A}$$
$$\Rightarrow x = f(a),\ y = f(a)\ \ [\textit{by def}^n \textit{ of inverse mapping}]$$
$$\Rightarrow x = y$$

So, by definition of one-one mapping it follows that $f^{-1} : \mathrm{B} \to \mathrm{A}$ is one-one.

Next, let, $a \in \mathrm{A}$, the co-domain of $f^{-1} : \mathrm{B} \to \mathrm{A}$, be arbitrary.

Now,

$$f : \mathrm{A} \to \mathrm{B}, a \in \mathrm{A} \Rightarrow \exists \textit{ an element } b \in \mathrm{B} \textit{ such that } f(a) = b$$
$$\Rightarrow \exists \textit{ an element } b \in \mathrm{B} \textit{ such that } f^{-1}(b) = a\ [\because f \textit{ is one-one}]$$

Hence $f^{-1} : \mathrm{B} \to \mathrm{A}$ is onto.

Thus $f^{-1} : \mathrm{B} \to \mathrm{A}$ is also a bijective mapping whenever $f : \mathrm{A} \to \mathrm{B}$ is bijective. #

Theorem: 2.6.02: ***The inverse of a one-one onto mapping is unique.***

Proof: Let $f : \mathrm{A} \to \mathrm{B}$ be one-one and onto. Then it possesses inverse.

We have to prove that the inverse of $f : \mathrm{A} \to \mathrm{B}$ is unique. If possible, let, there be two inverses $g : \mathrm{B} \to \mathrm{A}$ and $h : \mathrm{B} \to \mathrm{A}$.

Then,

$$g : \mathrm{B} \to \mathrm{A},\ h : \mathrm{B} \to \mathrm{A},\ b \in \mathrm{B} \Rightarrow g(b) = a_1(say) \in \mathrm{A},\ h(b) = a_2(say) \in \mathrm{A} \ldots\ldots\ldots\ldots(1)$$
$$\Rightarrow f(a_1) = b,\ f(a_2) = b \qquad [\because g, h \textit{ are inverses of } f]$$
$$\Rightarrow f(a_1) = b = f(a_2)$$
$$\Rightarrow a_1 = a_2 \qquad [\because f \textit{ is one-one}]$$
$$\Rightarrow g(b) = h(b),\ \forall b \in \mathrm{B} \qquad [\textit{ from } (1)]$$
$$\Rightarrow g = h \qquad [\textit{from def}^n \textit{ of equality of fns.}]$$
$$\Rightarrow f^{-1} \textit{ is unique.} \qquad \oplus$$

Theorem: 2.6.03***: If*** $f : \mathrm{A} \to \mathrm{B}$ ***and*** $g : \mathrm{B} \to C$ ***be one-one and onto, then***

(i) $f^{-1} \circ f = \mathrm{I}_\mathrm{A}$ *and* $f \circ f^{-1} = \mathrm{I}_\mathrm{B}$

(ii) $g \circ f : \mathrm{A} \to C$ *is one-one onto and* $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof:

(i)

$$f:\mathrm{A}\to\mathrm{B}\ \textit{is one-one and onto} \Rightarrow f^{-1} \textit{exists and } f^{-1}:\mathrm{B}\to\mathrm{A}\ \textit{is } 1-1\ \textit{onto}$$
$$\Rightarrow f^{-1}\circ f:\mathrm{A}\to\mathrm{A},\ f\circ f^{-1}:\mathrm{B}\to\mathrm{B}$$
$$\textit{Now, } (f^{-1}\circ f)(a) = f^{-1}(f(a)) = f^{-1}(b)\quad \textit{where } f(a)=b$$
$$= a \qquad [\because f(a)=b,\ f\ \textit{is } 1-1 \Rightarrow a = f^{-1}(b)]$$
$$= \mathrm{I}_{\mathrm{A}}(a)\quad \forall a\in\mathrm{A}$$
$$\textit{Therefore, } f^{-1}\circ f = \mathrm{I}_{\mathrm{A}}$$

Similarly, we can easily prove that $f\circ f^{-1} = \mathrm{I}_{\mathrm{B}}$

(ii) $f:\mathrm{A}\to\mathrm{B},\ g:\mathrm{B}\to C \Rightarrow g\circ f:\mathrm{A}\to C$

Now we have,

$$(g\circ f)(x) = (g\circ f)(y) \Rightarrow g(f(x)) = g(f(y))$$
$$\Rightarrow f(x) = f(y) \qquad [\because g\ \textit{is } 1-1]$$
$$\Rightarrow x = y \qquad [\because f\ \textit{is } 1-1]$$
$$\Rightarrow g\circ f\ \textit{is one-one}$$

Again, let $z\in C$ be arbitrary. Then, we have,

$$z\in C\ \&\ g:\mathrm{B}\to C\ \textit{is onto} \Rightarrow \exists y\in\mathrm{B}\ s.t.\ g(y)=z$$
$$\Rightarrow \exists x\in\mathrm{A}\ s.t.\ f(x)=y\ \textit{ where } g(y)=z\ [\because f:\mathrm{A}\to\mathrm{B}\ \textit{is onto}]$$
$$\Rightarrow \exists x\in\mathrm{A}\ s.t.\ g[f(x)] = g(y) = z$$
$$\Rightarrow \exists x\in\mathrm{A}\ s.t.\ (g\circ f)(x) = z$$
$$\Rightarrow x\in\mathrm{A}\ \textit{is the pre-image of } z\in C\ \textit{under } g\circ f$$

Therefore, $g\circ f$ is onto.

Also,

$$f:\mathrm{A}\to\mathrm{B},\ g:\mathrm{B}\to C\ \textit{are bijective} \Rightarrow g\circ f:\mathrm{A}\to C\ \textit{is bijective}$$
$$\Rightarrow (g\circ f)^{-1}\ \textit{exists and } (g\circ f)^{-1}: C\to\mathrm{A}$$
$$\textit{And } f:\mathrm{A}\to\mathrm{B},\ g:\mathrm{B}\to\mathrm{A}\ \textit{are bijective} \Rightarrow f^{-1}:\mathrm{B}\to\mathrm{A},\ g^{-1}:C\to\mathrm{B}\ \textit{exist}$$
$$\Rightarrow f^{-1}\circ g^{-1}: C\to\mathrm{A}$$
$$\therefore\ \textit{Domain and codomain of } (g\circ f)^{-1}\ \textit{and } f^{-1}\circ g^{-1}\ \textit{are same}.$$

Further, let $z \in C$ be arbitrary. Then,

$$z \in C \ and \ g : \mathrm{B} \to C \ is \ bijective \Rightarrow \exists\, y \in \mathrm{B} \ s.t. \ g(y) = z \ and \ so \ y = g^{-1}(z)$$
$$And \ \ y \in \mathrm{B}, \ f : \mathrm{A} \to \mathrm{B} \ is \ bijective \Rightarrow \exists\, x \in \mathrm{A} \ s.t. \ f(x) = y \ and \ so \ x = f^{-1}(y)$$

Therefore, $(g \circ f)(x) = g[f(x)] = g(y) = z$ and hence $(g \circ f)^{-1}(z) = x$

And, $(f^{-1} \circ g^{-1})(z) = f^{-1}[g^{-1}(z)] = f^{-1}(y) = x$

Thus $(g \circ f)^{-1}(z) = x = (f^{-1} \circ g^{-1})(z), \forall z \in C$

So, from the definition of equality of functions it follows that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Theorem: 2.6.04: *Let* $f : \mathrm{X} \to \mathrm{Y}$ *and* $\{\mathrm{A}_\lambda : \lambda \in \Lambda\}$, $\{\mathrm{B}_\alpha : \alpha \in \Delta\}$ *be arbitrary collections of subsets of X and Y respectively. Then,*

(i) $f\left(\bigcup_{\lambda \in \Lambda} \mathrm{A}_\lambda\right) = \bigcup_{\lambda \in \Lambda} f(\mathrm{A}_\lambda)$ (iii) $f^{-1}\left(\bigcup_{\alpha \in \Delta} \mathrm{B}_\alpha\right) = \bigcup_{\alpha \in \Delta} f^{-1}(\mathrm{B}_\alpha)$

(ii) $f\left(\bigcap_{\lambda \in \Lambda} \mathrm{A}_\lambda\right) \subset \bigcap_{\lambda \in \Lambda} f(\mathrm{A}_\lambda)$ (iv) $f^{-1}\left(\bigcap_{\alpha \in \Delta} \mathrm{B}_\alpha\right) = \bigcap_{\alpha \in \Delta} f^{-1}(\mathrm{B}_\alpha)$

Proof: (i) We have,

$$y \in f\left(\bigcup_{\lambda \in \Lambda} \mathrm{A}_\lambda\right) \Rightarrow y = f(x) \ for \ some \ x \in \bigcup_{\lambda \in \Lambda} \mathrm{A}_\lambda$$
$$\Rightarrow y = f(x) \ where \ x \in \mathrm{A}_\lambda \ \ for \ some \ \lambda \in \Lambda$$
$$\Rightarrow y = f(x) \in f(\mathrm{A}_\lambda) \ \ for \ some \ \lambda \in \Lambda \ [\because x \in \mathrm{A}_\lambda \Rightarrow f(x) \in f(\mathrm{A}_\lambda)\ ]$$
$$\Rightarrow y = f(x) \in f(\mathrm{A}_\lambda) \subset \bigcup_{\lambda \in \Lambda} f(\mathrm{A}_\lambda)$$

$$\therefore \ f\left(\bigcup_{\lambda \in \Lambda} \mathrm{A}_\lambda\right) \subseteq \bigcup_{\lambda \in \Lambda} f(\mathrm{A}_\lambda) \text{..(1)}$$

Conversely,

$$y \in \bigcup_{\lambda \in \Lambda} f(\mathrm{A}_\lambda) \Rightarrow y = f(\mathrm{A}_\lambda) \ \ for \ some \ \lambda \in \Lambda$$
$$\Rightarrow y = f(x) \ \ for \ some \ \ x \in \mathrm{A}_\lambda \subset \bigcup_{\lambda \in \Lambda} \mathrm{A}_\lambda$$
$$\Rightarrow y = f(x) \in f(\mathrm{A}_\lambda) \subset f\left(\bigcup_{\lambda \in \Lambda} \mathrm{A}_\lambda\right) \ \ [\because x \in \mathrm{A}_\lambda \Rightarrow f(x) \in f(\mathrm{A}_\lambda)\ ]$$

$$\therefore \bigcup_{\lambda\in\Lambda} f(A_\lambda) \subseteq f\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) \text{...} (2)$$

From (1) and (2) it follows that $\bigcup_{\lambda\in\Lambda} f(A_\lambda) = f\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)$.

(ii)We have,

$$y \in f\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \Rightarrow y = f(x) \ \textit{for some} \ x \in \bigcap_{\lambda\in\Lambda} A_\lambda$$
$$\Rightarrow y = f(x) \ \textit{where} \ x \in A_\lambda, \forall \lambda \in \Lambda$$
$$\Rightarrow y = f(x) \ \textit{where} \ f(x) \in f(A_\lambda), \forall \lambda \in \Lambda$$
$$\Rightarrow y = f(x) \ \textit{where} \ f(x) \in \bigcap_{\lambda\in\Lambda} f(A_\lambda)$$

$$\therefore \ f\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \subset \bigcap_{\lambda\in\Lambda} f(A_\lambda)$$

In general, the equality doesn't hold.

For example, consider the mapping $f : R \to R \ \ s.t. \ \ f(x) = x^2$ and $A_1 = [-1,0], \ A_2 = [0,1]$

Then, $A_1 \cap A_2 = [-1,0] \cap [0,1] = \{0\} \Rightarrow f(A_1 \cap A_2) = f(\{0\}) = \{0\}$

And $f(A_1) \cap f(A_2) = f([-1,0]) \cap f([0,1]) = [0,1] \cap [0,1] = [0,1] \neq f(A_1 \cap A_2)$

(iii) We have,

$$x \in f^{-1}\left[\bigcup_{\alpha\in\Delta} B_\alpha\right] \Leftrightarrow f(x) \in \bigcup_{\alpha\in\Delta} B_\alpha \Leftrightarrow f(x) \in B_\alpha \ \ \textit{for some} \ \ \alpha \in \Delta$$
$$\Leftrightarrow x \in f^{-1}(B_\alpha) \ \ \textit{for some} \ \ \alpha \in \Delta$$
$$\Leftrightarrow x \in \bigcup_{\alpha\in\Delta} f^{-1}(B_\alpha)$$

$$\therefore \quad f^{-1}\left[\bigcup_{\alpha\in\Delta} B_\alpha\right] = \bigcup_{\alpha\in\Delta} f^{-1}(B_\alpha)$$

(iv) We have,

$$x \in f^{-1}\left[\bigcap_{\alpha\in\Delta} B_\alpha\right] \Leftrightarrow f(x) \in \bigcap_{\alpha\in\Delta} B_\alpha \Leftrightarrow f(x) \in B_\alpha \quad for\ \ every\ \ \alpha \in \Delta$$

$$\Leftrightarrow x \in f^{-1}(B_\alpha) \quad for\ \ every\ \ \alpha \in \Delta$$

$$\Leftrightarrow x \in \bigcap_{\alpha\in\Delta} f^{-1}(B_\alpha)$$

$$\therefore \quad f^{-1}\left[\bigcap_{\alpha\in\Delta} B_\alpha\right] = \bigcap_{\alpha\in\Delta} f^{-1}(B_\alpha)$$

Theorem: 2.6.05: *Let* $f : X \to Y$ *and* $A, C \subset X$, $B, D \subset Y$ *be arbitrary. Then,*

(i) $f(A) = B \Rightarrow A \subseteq f^{-1}(B)$*, the equality holds when* f *is one–one*

(ii) $A = f^{-1}(B) \Rightarrow f(A) \subseteq B$*, the equality holds when* f *is onto*

(iii) $A \subset C \Rightarrow f(A) \subset f(C)$

(iv) $B \subset D \Rightarrow f^{-1}(B) \subset f^{-1}(D)$

(v) $A \subseteq f^{-1}[f(A)]$*, the equality holds when* f *is one–one*

(vi) $f[f^{-1}(B)] \subseteq B$*, the equality holds when* f *is onto*

(vii) $f^{-1}(B') = [f^{-1}(B)]'$

(viii) $f^{-1}(B - D)] = f^{-1}(B) - f^{-1}(D)$

Proof:

(i) Let $f(A) = B$. Then,

$$x \in A \Rightarrow f(x) \in f(A) = B\,(given)$$

$$\Rightarrow x \in f^{-1}(B)$$

$$\therefore \quad A \subseteq f^{-1}(B)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

To prove the converse part for equality, let *f* be one-one. Then,

$$x \in f^{-1}(B) \Rightarrow f(x) \in B = f(A)\ (given)$$

$$\Rightarrow f(x) = f(a) \quad for\ \ some\ \ a \in A$$

$$\Rightarrow x = a \qquad [\because f\ \ is\ 1-1]$$

$$\therefore \quad f^{-1}(B) \subseteq A\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(b)$$

As a consequence of (a) and (b), we get, $A = f^{-1}(B)$ whenever f is one-one.

(ii) Let $A = f^{-1}(B)$. Then,

$$y \in f(A) \Rightarrow y = f(x) \in f(A) \quad for \ \ some \ \ x \in A$$
$$Now, \ \ x \in A \Rightarrow x \in f^{-1}(B) \quad [\because A = f^{-1}(B)] \Rightarrow f(x) = y \in B$$
$$\therefore \quad f(A) \subseteq B....................................(a)$$

To prove the converse part for equality, let *f* be onto. Then,

$$y \in B \Rightarrow \exists x \in A \ s.t. \ y = f(x)$$
$$\Rightarrow f(x) \in f(A)$$
$$\Rightarrow y \in f(A)$$
$$\therefore \quad B \subseteq f(A).....................................(b)$$

As a consequence of (a) and (b), we get, $B = f(A)$ whenever f is one-one.

(iii) We have to prove that $A \subset C \Rightarrow f(A) \subset f(C)$

We have,

$$y \in f(A) \Rightarrow \exists x \in A \ s.t. \ f(x) = y$$
$$\Rightarrow \exists x \in C \ s.t. \ f(x) = y \ [\because A \subseteq C]$$
$$\Rightarrow y = f(x) \in f(C) \quad [\because x \in C \Rightarrow f(x) \in f(C)]$$
$$\therefore \quad f(A) \subset f(C)$$

(iv) Let $B \subset D$. Then, we have,

$$x \in f^{-1}(B) \Rightarrow \exists y \in B \ s.t. \ f(x) = y$$
$$\Rightarrow \exists y \in D \ s.t. \ f(x) = y \quad [\because B \subseteq D]$$
$$\Rightarrow f(x) \in D \quad [\because f(x) = y]$$
$$\Rightarrow x \in f^{-1}(D)$$
$$\therefore \quad f^{-1}(B) \subset f^{-1}(D)$$

(v) We have,

$$x \in A \Rightarrow f(x) \in f(A) \Rightarrow x \in f^{-1}[f(A)]$$
$$Thus \ \ x \in A \Rightarrow x \in f^{-1}[f(A)]$$
$$\therefore \quad A \subseteq f^{-1}[f(A)]$$

If f is one-one, then,

$$x \in f^{-1}[f(A)] \Rightarrow f(x) \in f(A)$$
$$\Rightarrow x \in A \quad [\because f \ \ is \ one-one]$$
$$\therefore \quad f^{-1}[f(A)] \subseteq A$$

So, in this case $A = f^{-1}[f(A)]$

To prove that $A = f^{-1}[f(A)]$ is not necessarily true, we again consider the mapping $f : R \to R$ such that $f(x) = x^2$ and $A = [-1, 0]$.

Then, $f(A) = [0,1]$ and clearly $f^{-1}[f(A)] = [-1,1] \neq A$.

(vi) We have,

$$y \in f[f^{-1}(\mathrm{B})] \Rightarrow y = f(x) \text{ for some } x \in f^{-1}(\mathrm{B})$$
$$\Rightarrow y = f(x) \text{ where } f(x) \in \mathrm{B}$$
$$\Rightarrow y \in \mathrm{B}$$
$$\therefore \quad f[f^{-1}(\mathrm{B})] \subset \mathrm{B}$$

In general, $\mathrm{B} \not\subset f[f^{-1}(\mathrm{B})]$. To establish this consider the mapping $f: R \to R$ such that $f(x) = x^2$ and $\mathrm{B} = [-1, 0]$.

Then, $f^{-1}[\mathrm{B}] = \{0\}$ and so $f[f^{-1}(\mathrm{B})] = f[\{0\}] = \{0\}$ and $[-1, 0] = \mathrm{B} \not\subset f[f^{-1}(\mathrm{B})]$.

But, if f is onto, then,

$$y \in \mathrm{B} \Rightarrow \exists x \in f^{-1}(B) \ s.t. \ f(x) = y$$
$$\Rightarrow y = f(x) \in f[f^{-1}(\mathrm{B})]$$
$$\therefore \quad \mathrm{B} \subseteq f[f^{-1}(\mathrm{B})]$$

So, in this case, $\mathrm{B} = f[f^{-1}(\mathrm{B})]$

(vii) We have,

$$x \in f^{-1}(\mathrm{B}') \Leftrightarrow f(x) \in \mathrm{B}'$$
$$\Leftrightarrow f(x) \notin \mathrm{B}$$
$$\Leftrightarrow x \notin f^{-1}(\mathrm{B})$$
$$\Leftrightarrow x \in [f^{-1}(\mathrm{B})]'$$
$$\therefore \quad f^{-1}(\mathrm{B}') = [f^{-1}(\mathrm{B})]'$$

(viii) Finally, we have,

$$x \in f^{-1}[\mathrm{B} - D] \Leftrightarrow f(x) \in \mathrm{B} - D$$
$$\Leftrightarrow f(x) \in \mathrm{B} \text{ and } f(x) \notin D$$
$$\Leftrightarrow x \in f^{-1}(\mathrm{B}) \text{ and } x \notin f^{-1}(D)$$
$$\Leftrightarrow x \in f^{-1}(\mathrm{B}) - f^{-1}(D)$$
$$\therefore \quad f^{-1}[\mathrm{B} - D] = f^{-1}(\mathrm{B}) - f^{-1}(D)$$

****