Example:- (Antibiotics) Tetracycline is an antibiotic prescribed for a range of problems from acne to acute infections. A course is taken orally and the drug moves from the GI-tract through the blood stream, from which it is removed by the kidneys and excreted in the urine.

a) Write word eqns to describe the movement of a drug through the body, using three compartments: the GI-tract, the bloodstream and the urinary tract. Note that the drug enters but is not removed from the urinary tract.

b) From the word eqns developed the diff. eqn system that describes this process defining all variables and parametres as required.

c) The constants of proportionality associated with the rates at which tetracycline (measured in milligrams) diffuses from the GI-tract into the bloodstream and then is removed, are 0·72 $hours^{-1}$ and 0·25 $hours^{-1}$ respectively (Borelli and Columan 1996). Suppose initially, the amount of tetracycline in the GI-tract is 0·0001 milligram, while there is none in the bloodstream or urinary tract.

Solve this system analytically and thus establish how the ~~tete~~ levels of tetracycline change with time in each of the compartments. In the case of a single dose establish the maximum level reached by the drug in the blood stream and how long it take to reach this level with the initial condⁿ as given above.

Solⁿ :→ Let, $u(t)$, $v(t)$ and $w(t)$ are the amount of drug in GI-tract, in bloodstream & in ~~uri~~ urinary tract.

$$\longrightarrow \underset{}{\overset{u(t)}{\boxed{\text{GI-tract}}}} \longrightarrow \overset{v(t)}{\boxed{\text{blood stream}}} \longrightarrow \overset{w(t)}{\boxed{\text{urinary tract}}}$$

(a) Word eqⁿs are

$$\left\{\begin{matrix}\text{rate of change of}\\ \text{drug in GI-tract}\end{matrix}\right\} = -\left\{\begin{matrix}\text{rate of drug}\\ \text{leaves GI-tract}\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{rate of chang of}\\ \text{drug in bloodstream}\end{matrix}\right\} = \left\{\begin{matrix}\text{rate of drug}\\ \text{enters}\\ \text{bloodstream}\end{matrix}\right\} - \left\{\begin{matrix}\text{rate of drug}\\ \text{leaves}\\ \text{bloodstream}\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{rate of change of}\\ \text{drug in urinary tract}\end{matrix}\right\} = \left\{\begin{matrix}\text{rate of drug}\\ \text{enters}\\ \text{urinary tract}\end{matrix}\right\}.$$

(b) Using balance law, the word eqns are —

$$\frac{du}{dt} = -m_1 u; \quad u(0) = u_0 \quad \text{——— (i)}.$$

$$\frac{dv}{dt} = m_1 u - m_2 v; \quad v(0) = 0 \quad \text{——— (ii)}.$$

$$\frac{dw}{dt} = m_2 v; \quad w(0) = 0 \quad \text{——— (iii)}.$$

where, m_1 & m_2 are rate constant & $m_1 \neq m_2$.

(c) Given that, $m_1 = 0.72\ hr^{-1}$, $m_2 = 0.15\ hr^{-1}$, $u_0 = 0.0001\ mg$.

$\because \frac{du}{dt} = -m_1 u \Rightarrow \frac{du}{u} = -m_1 dt$.

Integrating, $\log u = -m_1 t + c_1$.

$\Rightarrow u(t) = e^{-m_1 t + c_1} = e^{c_1} \cdot e^{-m_1 t}$.

Using $u(0) = u_0$, $u_0 = e^{c_1}$.

$\therefore u(t) = u_0 e^{-m_1 t} = 0.0001 \cdot e^{-0.72t}$.

$\Rightarrow u(t) = 0.0001\, e^{-0.72t}$

Putting the value of u in (ii), we have—

$$\frac{dv}{dt} = -m_1 u_0 e^{-m_1 t} - m_2 v.$$

$$\Rightarrow \frac{dv}{dt} + m_2 v = m_1 u_0 e^{-m_1 t}.$$

I.F. $= e^{m_2 t}$.

$$\therefore v e^{m_2 t} = \int m_1 u_0 e^{-m_1 t} \cdot e^{m_2 t} dt + c_2$$

$$\Rightarrow v e^{m_2 t} = m_1 u_0 \frac{e^{(m_2 - m_1)t}}{m_2 - m_1} + c_2.$$

Using, $v(0) = 0$, we get $c_2 = -\frac{m_1 u_0}{m_2 - m_1}$.

$$\therefore \cancel{vt}\ v = \frac{m_1 u_0}{m_2 - m_1}\left(e^{-m_1 t} - e^{-m_2 t}\right) \text{ — (iv)}.$$

Putting the value of m_1, m_2 & u_0 in (iv), we get

$$V = \frac{0.0001 \times 0.72}{0.15 - 0.72}\left(e^{-0.72t} - e^{-0.15t}\right)$$

$$= -0.00012\left(e^{-0.72t} - e^{-0.15t}\right)$$

$$= 0.00012\left(e^{-0.15t} - e^{-0.72t}\right)$$

Putting the value of v from eqn (iv) in eqn (iii),

$$\frac{dw}{dt} = m_2 \frac{m_1 u_0}{m_2 - m_1}\left(e^{-m_1 t} - e^{-m_2 t}\right).$$

$$\Rightarrow dw = \frac{m_1 m_2 u_0}{m_2 - m_1}\left(e^{-m_1 t} - e^{-m_2 t}\right) dt$$

Integrating,

$$w(t) = \frac{m_1 m_2 u_0}{m_2 - m_1}\left(\frac{e^{-m_1 t}}{-m_1} - \frac{e^{-m_2 t}}{-m_2}\right) + e_3 .$$

$$= \frac{m_1 m_2 u_0}{m_2 - m_1}\left(\frac{e^{-m_2 t}}{m_2} - \frac{e^{-m_1 t}}{m_1}\right) + e_3 \quad \text{— (v).}$$

Using $w(0) = 0$, $e_3 = -\frac{m_1 m_2 u_0}{m_2 - m_1}\left(\frac{1}{m_2} - \frac{1}{m_1}\right)$.

$$\Rightarrow e_3 = -\frac{m_1 m_2 u_0}{m_2 - m_1} \times \frac{m_1 - m_2}{m_1 m_2} = u_0 .$$

Put $e_3 = u_0$ in (v),

$$w(t) = \frac{m_1 m_2 u_0}{m_2 - m_1}\left(\frac{e^{-m_2 t}}{m_2} - \frac{e^{-m_1 t}}{m_1}\right) + u_0 .$$

Putting the value of m_1, m_2 & u_0 in this eqn,

$$w(t) = \frac{0.72 \times 0.15 \times 0.0001}{0.15 - 0.72}\left(\frac{e^{-0.15t}}{0.15} - \frac{e^{-0.72t}}{0.72}\right) + 0.0001$$

$$= 0.00002\left(\frac{e^{-0.15t}}{0.15} - \frac{e^{-0.72t}}{0.72}\right) + 0.0001$$