3. Lake Pollution Model :- Lake pollution model is concerned with the pollution in the lake and rivers that has become a major problem particularly over the last 50 years. This model can be considered as a compartmental model with a single compartment, the lake. It can be represented in the form of the following compartmental diagram.

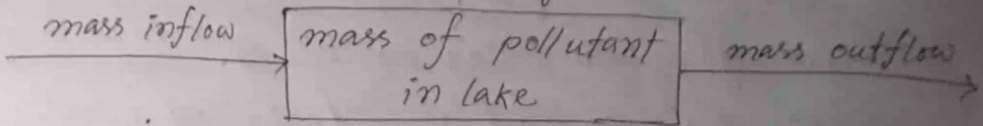

fig: Input-output compartmental diagram for lake pollution.

By balance law, the word eq^n for the mass of pollution in lake

$$\begin{Bmatrix}\text{rate of change} \\ \text{of mass in lake}\end{Bmatrix} = \begin{Bmatrix}\text{rate mass} \\ \text{enters lake}\end{Bmatrix} - \begin{Bmatrix}\text{rate mass} \\ \text{leaves lake}\end{Bmatrix}$$

Formulation the differential eq^n :- We assume that volume of lake is constant and that it is continuously well mixed so that the pollution is uniform throughout.

Let, $c(t)$ = the concentration of the ~~pollution~~ pollutant in the lake at time t and f = the rate of which water flow out of the lake in m^3/day.

Since volume is constant, we have–

$$\begin{Bmatrix}\text{flow of mixture} \\ \text{into lake}\end{Bmatrix} = \begin{Bmatrix}\text{flow of mixture} \\ \text{out of lake}\end{Bmatrix} = f.$$

Applying the balance law to the mass of pollutant $m(t)$, word eq^n is given by-

$$\begin{Bmatrix}\text{rate of change of}\\ \text{mass of pollutant}\\ \text{in lake}\end{Bmatrix} = \begin{Bmatrix}\text{rate at which}\\ \text{the pollutant}\\ \text{enters the lake}\end{Bmatrix} - \begin{Bmatrix}\text{rate at which}\\ \text{the pollutant}\\ \text{leaves the lake}\end{Bmatrix}$$

i.e. $\frac{dm(t)}{dt} = f c_{in} - f \frac{m(t)}{V}$ —— ①.

where V is the volume of lake and c_{in} is the concentration (in unit mass per unit of volume, such as g/m^3) of the pollutant in the flow entering the lake.

$\because \quad m(t) = c(t) V$

$\therefore \quad m'(t) = c'(t) V$ (since V is constant).

$\Rightarrow c'(t) = \frac{m'(t)}{V}$.

Now, eq^n ① becomes, $V \cdot \frac{dc(t)}{dt} = f c_{in} - f \frac{c(t) V}{V}$.

$$\Rightarrow \frac{dc}{dt} = \frac{f}{V} c_{in} - \frac{f}{V} c$$

$$\Rightarrow \frac{dc}{dt} = \frac{f}{V} (c_{in} - c).$$

This is the differential eq^n of the lake pollution model.

Example: 1. Solve the d. eq^n, $\frac{dc}{dt} = \frac{f}{V}(c_{in} - c)$ with the initial cond^n $c(0) = c_0$.

Sol^n:- Given that, $\frac{dc}{dt} = \frac{f}{V}(c_{in} - c)$

$$\Rightarrow \frac{dc}{(c_{in} - c)} = \frac{f}{V} dt$$

Integrating, $-\log|c_{in} - c| = \frac{f}{V} t + A$; where A is constant of integration.

$$\Rightarrow -\log|c_{in} - c| = -\frac{f}{V} t - A$$

$$\Rightarrow c_{in} - c = e^{\frac{f}{V}t - A} = e^{-A} e^{-\frac{f}{V}t}$$

$$\Rightarrow c(t) = c_{in} - e^{-A} e^{-f/V t} = c_{in} - B e^{-\frac{f}{V}t} \text{ —— ①.}$$

where $e^{-A} = B$.

using initial cond^n, $c(0) = c_0$, we get—

$$C_0 = C_{in} - B \Rightarrow B = C_{in} - C_0.$$

from (1), $$c(t) = C_{in} - (C_{in} - C_0)\, e^{-f/v\, t}$$

Note:- The sol^n can be divided into two parts: $C_0 e^{-f/v\, t}$ is the contribution from the initial data and $C_{in} - C_{in} e^{-f/v\, t}$ is the contribution from the pollution inflow to the system,

$c(t) \to C_{in}$ as $t \to \infty$.

i.e. $$\lim_{t\to\infty}\left[C_{in} - (C_{in} - C_0)\, e^{-\frac{f}{v}t}\right] = C_{in}.$$

Hence, the concentration in the lake increases/ decreases steadily to this limit.

Example: 2 Let in a lake the pollution level is 7%. If the concentraction of the incoming water is 2% and 10,000 litres per day water is allowed to enter the lake, find time when pollution level is 5% and volume of the lake is 2,00,000 litres. Also find the pollution after 32 days.

Sol^n:- Given, $V = 200000$, $f = 10000$, $C_{in} = 0.02$, $c(t) = 0.05$, $C_0 = 0.07$,

$$\because\ c(t) = C_{in} - (C_{in} - C_0)\, e^{-\frac{f}{v}t} \quad \text{—(1).}$$

$$\Rightarrow e^{-\frac{f}{v}t} = \frac{C_{in} - c(t)}{C_{in} - C_0}$$

$$\Rightarrow e^{-\frac{10000}{200000}t} = \frac{0.02 - 0.05}{0.02 - 0.07} = \frac{-0.03}{-0.05} = \frac{3}{5}.$$

$$\Rightarrow e^{-\frac{1}{20}t} = 0.6$$

$$\Rightarrow \log 0.6 = -\frac{1}{20}t \Rightarrow -\frac{1}{20}t = -0.5107$$

$$\Rightarrow t = 10.214 \text{ days}.$$

Hence, the reqd time = 10·214 days.

Put, $t = 32$ in (i), we have—

$$c(t) = c_{in} - (c_{in} - c_0)\, e^{-\frac{f}{V} 32}$$

$$= 0{\cdot}02 - (0{\cdot}02 - 0{\cdot}07)\, e^{-\frac{10000}{200000} \times 32}$$

$$= 0{\cdot}02 + 0{\cdot}05\, e^{-1{\cdot}6}$$

$$= 0{\cdot}02 + 0{\cdot}05\,(0{\cdot}2019)$$

$$= 0{\cdot}030095.$$

∴ pollution level after 32 days = 3% approx. //

<u>Example: 3</u> How long will it take for the lake's pollution level to reach 5% of its initial level, if only fresh water flow into the lake.

<u>Soln</u> :→ We know that, $c(t) = c_{in} - (c_{in} - c_0)\, e^{-\frac{f}{V} t}$ —— (i).

Since, only freshwater flow into the lake, so $c_{in} = 0$.

From (i), $c(t) = c_0\, e^{-\frac{f}{V} t}$.

$$\Rightarrow \log c(t) = \log c_0 - \frac{f}{V} t \log e.$$

$$\Rightarrow \frac{ft}{V} = \log c_0 - \log c(t).$$

$$\Rightarrow t = \frac{V}{f} \log \frac{c_0}{c(t)} \quad \text{—— (ii)}.$$

Given that, $c(t) = 5\%$ of $c_0 = \frac{5}{100} c_0 = 0{\cdot}05 c_0$.

From (ii),

$$t = \frac{V}{f} \log \frac{c_0}{0{\cdot}05 c_0} = \frac{V}{f} \log 20 = \frac{3V}{f}.$$

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