Unit-2

Introduction to compartmental Models :-

The compartmental model is a model for process that have inputs and/or outputs over time. In this chapter, we model exponential decay model, pollution levels in lake systems, drug assimulation in to the blood, exponential growth include the effect of population growth of limited resources using compartmental techniques. We discuss the effect of harvesting the population and see that there is a critical harvesting rate.

1. Compartmental Model :- compartmental model is a model in which there is a place called compartment which has amount of substance in and out over time. One example of compartmental model is the amount of CO_2 in the earth's atmosphere. The compartment is atmosphere where the input of CO_2 occurs through many processes such as burning and output of CO_2 occurs through plant respiration.

$$\xrightarrow[CO_2]{\text{Input}} \boxed{\text{Atmosphere}} \xrightarrow[CO_2]{\text{Output}}$$

fig.: Input-Output Compartmental Diagram for CO_2.

Balance law :- Balance law starts that the rate of change of quantity of substance is equal to 'rate in' minus 'rate out' of the compartment.

i.e. $\begin{Bmatrix} \text{Net rate of change} \\ \text{of a substance} \end{Bmatrix} = \{\text{rate in}\} - \{\text{rate out}\}$.

Which is known as word eqⁿ of the model.

2. Exponential Decay Model :-

Radioactive elements are those elements which are not stable and emit α- particles, β particles or photons while decaying into isotopes of other elements. Exponential decay model for radioactive decay can be considered as a compartmental model with compartment being the radioactive meterial with no input but output as decay of radioactive sample over time.

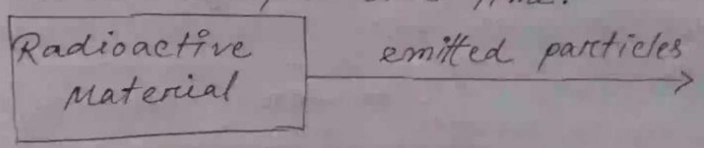

fig:- Input - output compartmental diagram for radio- -active nuclei.

By Balance law, word eqⁿ is given by,

$$\begin{Bmatrix} \text{Rate of change of radioactive} \\ \text{material at time } t \end{Bmatrix} = -\begin{Bmatrix} \text{rate amount of} \\ \text{radioactive} \\ \text{material decayed} \end{Bmatrix}$$

We make the following assumption :

* We assume that the amount of an element present in large enough so that we are justified in ignoring random fluctuations.
* We assume the process is continuous in time.
* We assume a fixed rate of decay for an element.
* Assume that there is no increase in mass of the body of material.

Formulation the differential Eqⁿ :-

- Let, $n(t)$ = no. of radioactive nuclei at time t.

and Δt = small change in time.

$\because$ the change in the no. of nuclei $\propto$ no. of nuclei at the start of time period.

$\Rightarrow \quad \frac{dn}{dt} \propto n.$

$\Rightarrow \quad \frac{dn}{dt} = -Kn$, where K is the +ve constant of proportionality indicating rate of decay per unelei in unit time.

At initial condⁿ, no. of radioactive nuclei = n_0 therefore $n(0) = n_0$.

Hence, IVP corresponding to exponential decay model is given by,

$$\boxed{\frac{dn}{dt} = -Kn \; ; \; n(0) = n_0}$$

Example :- 1. Solve the initial value problem (IVP)

$$\frac{dn}{dt} = -Kn \text{ with initial condition } n(t_0) = n_0.$$

Solⁿ : $\frac{dn}{dt} = -Kn \Rightarrow \frac{dn}{n} = -K\,dt$

Integrating, $\log n = -Kt + c$ —— ①.

Where c is integrating constant.

From ①, $n(t) = e^{-Kt+c} = e^{-Kt} \cdot e^{c}$.

$\Rightarrow n(t) = Ae^{-Kt}$, where $A = e^{c}$.

using, $n(t_0) = n_0$, we get —

$$n_0 = Ae^{-Kt_0} \Rightarrow A = n_0 e^{Kt_0}$$

$$\therefore n(t) = n_0 e^{Kt_0} e^{-Kt} = n_0 e^{-K(t-t_0)}$$

$$\Rightarrow n(t) = n_0 e^{-K(t-t_0)}$$

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Example:2 : Radium decomposes at a rate proportional to the amount present. If half the original amount disappears in 1600 years, find the percentage lost in 100 years.

Soln:- Let, $m(t)$ = the mass at time t (years), then IVP of decay model is given by,

$$\frac{dm}{dt} = -km \quad ; \quad m(0) = m_0 .$$

$$\Rightarrow \quad \frac{dm}{m} = -k\,dt .$$

Integrating, $\log m = -kt + c$ —— ①.

where c is constant of integrating.

$\therefore$ ① $\Rightarrow$ $m = e^{-kt+c} = Ae^{-kt}$; where $e^c = A$.

If $m(0) = m_0$, we get— $m_0 = Ae^{-k.0} = A$.

Hence, $m = m_0 e^{-kt}$ —— ⓘⓘ.

Given, $m = \frac{1}{2} m_0$, when $t = 1600$.

$\therefore$ $\frac{1}{2} m_0 = m_0 e^{-kt} \Rightarrow 2 = e^{1600k}$ —— ⓘⓘⓘ.

when, $t = 100$, $m = m_0 e^{-100k}$; (from ⓘⓘ)

$$= m_0 \cdot \left(e^{1600k}\right)^{-1/16} = m_0 \cdot (2)^{-1/16}$$

$$\therefore m = \frac{m_0}{(2)^{1/16}} = 0.96\, m_0 = 96\% \text{ of } m_0 .$$

Hence, loss $= (100 - 96)\% = 4\%$.