Unit - 8 (General) 4th Sem

1. Vector Point function :– If R represents the real numbers set and V represents the vector space and let $[a,b] \subset R$, then the function, $\vec{f} : [a,b] \to V$ s.t. $\vec{f}(t) = \vec{r} \in V$, where $t \in [a,b]$, is called vector point function of scalar variable t.

Remark :- If $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$; where f_1, f_2, f_3 are scalar components of f in the direction of $\hat{i}, \hat{j}, \hat{k}$.

we have, $\vec{f}(t) = \vec{r}$

$$\Rightarrow (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})(t) = x\hat{i} + y\hat{j} + z\hat{k} \,;$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$$\Rightarrow f_1(t) = x,\ f_2(t) = y,\ f_3(t) = z.$$

2. Limit of a vector point function :- A vector point function $\vec{f}(t)$ where, $\vec{f} : R \to V$ is said to tend to a limit $\vec{l}$, as t tends to 'a' $\in R$, if any pre-assigned positive no. ε, however small, $\exists$ a +ve no. δ, such that

$$|\vec{f}(t) - \vec{l}\,| < \varepsilon \,, \text{ when } 0 < |t-a| \le \delta.$$

Symbolically, we write $\lim_{t\to a} \vec{f}(t) = \vec{l}$.

3. Continuity of a vector point function :-

A vector function $\vec{f}(t)$ of a scalar variables t is said to be continuous at $t = a$, if

(i) $\vec{f}(a)$ is defined / exists

(ii) $\lim_{t\to a} \vec{f}(t) = \vec{f}(a)$.

Theorem :-

1. If $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ be continuous, then $f_1(t), f_2(t), f_3(t)$ are also continuous.

2. If $\vec{f}(t)$ and $\vec{g}(t)$ be two continuous vector functions and $\phi(t)$ be a continuous scalar function of t, then,

a) $\vec{f}(t) \pm \vec{g}(t)$ is also continuous.

b) $\vec{f}(t) \cdot \vec{g}(t)$ " " " .

c) $\vec{f}(t) \times \vec{g}(t)$ " " " .

d) $\phi(t) \cdot \vec{f}(t)$ " " " .

4. Derivative of a vector function :-

If $\vec{f}(t)$ be a vector function of the scalar variable t, then, $\lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t}$ if exists, is called the derivative of $\vec{f}(t)$ w.r.t. t and denoted by, $\frac{d\vec{f}}{dt}$ or $\vec{f}'(t)$ etc.

Higher Order Derivatives :-

If differentiate $\frac{d\vec{f}}{dt}$ w.r.t. 't' then it is denoted by, $\frac{d^2\vec{f}}{dt^2}$ or $\vec{f}''(t)$ and is called the second order derivative of $\vec{f}(t)$.

Similarly, we have—

$$\frac{d^3\vec{f}}{dt^3}, \frac{d^4\vec{f}}{dt^4}, \ldots, \frac{d^n\vec{f}}{dt^n} \text{ etc.}$$

Derivatives in terms of components :-

Let, $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ and

$\vec{f}(t+\Delta t) = f_1(t+\Delta t)\hat{i} + f_2(t+\Delta t)\hat{j} + f_3(t+\Delta t)\hat{k}$

then,

$$\Delta\vec{f} = \vec{f}(t+\Delta t) - \vec{f}(t).$$

$$= [f_1(t+\Delta t)\hat{i} + f_2(t+\Delta t)\hat{j} + f_3(t+\Delta t)\hat{k}] - [f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}]$$

$$= [f_1(t+\Delta t) - f_1(t)]\hat{i} + [f_2(t+\Delta t) - f_2(t)]\hat{j} + [f_3(t+\Delta t) - f_3(t)]\hat{k}.$$

$$= \Delta f_1\hat{i} + \Delta f_2\hat{j} + \Delta f_3\hat{k}. \text{ (say)}.$$

and $$\frac{\Delta\vec{f}}{\Delta t} = \frac{\Delta f_1}{\Delta t}\hat{i} + \frac{\Delta f_2}{\Delta t}\hat{j} + \frac{\Delta f_3}{\Delta t}\hat{k}.$$

Taking, $\Delta t \to 0$, we get —

$$\frac{d\vec{f}}{dt} = \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}.$$

Properties of differentiations :-

Let, $\vec{f}(t)$ and $\vec{g}(t)$ be two derivable vector functions and $\phi(t)$ be a derivable scalar, function of t then,

(i) If $\vec{f}(t)$ is constant, $\Rightarrow \frac{d\vec{f}}{dt} = 0$.

(ii) $\frac{d}{dt}[\vec{f} \pm \vec{g}] = \frac{d\vec{f}}{dt} \pm \frac{d\vec{g}}{dt}$

(iii) $\frac{d}{dt}[\vec{f}\cdot\vec{g}] = \vec{f}\cdot\frac{d\vec{g}}{dt} + \frac{d\vec{f}}{dt}\cdot\vec{g}$.

(iv) $\frac{d}{dt}[\vec{f}\times\vec{g}] = \vec{f}\times\frac{d\vec{g}}{dt} + \frac{d\vec{f}}{dt}\times\vec{g}$.

(v) $\frac{d}{dt}[\phi\cdot\vec{f}] = \phi\cdot\frac{d\vec{f}}{dt} + \vec{f}\frac{d\phi}{dt}$.