Compound Interest :- Let P be the principal amount in an account at time t (in years) at the rate of interest r compound continuously. Then it can be modelled as a differential eqn as,

$$\frac{dP}{dt} = rP \Rightarrow \frac{dP}{P} = r\,dt. \Rightarrow \int \frac{dP}{P} = \int r\,dt$$

$$\Rightarrow \log P = rt + c \quad \text{--- (1)}$$, where c is an integrating constant.

Let the amount deposited in the account initially be P_0. Then at $t=0$, $P = P_0$, from (1), $\log P_0 = c$.

Therefore, the amount in the account at the end of t years compounded continuously at the rate of interest r is $P = P_0 e^{rt}$.

Radioactive Decay :- Let, $N(t)$ be the no. of atoms of a certain radioactive isotope at time t.

A constant fraction of those radioactive atoms decay spontaneously into atoms of another element or into another isotope of the same element during each unit of time. It can be modelled as,

$$\frac{dN}{dt} = -\lambda N,$$ where λ is a constant of proportionality.

$$\Rightarrow \frac{dN}{N} = -\lambda\,dt \Rightarrow \log N = -\lambda t + c \quad \text{--- (1)}.$$

where c is the constant of integration.

Let, the no. of atoms be N_0 at $t=0$. Then

① $\Rightarrow \log N_0 = c$.

Thus, ① becomes, $\log N = -\lambda t + \log N_0$

$$\Rightarrow N = N_0 e^{-\lambda t}.$$

The value of decay constant κ depends on the particular radioactive isotope for ^{14}C, it is known that $\lambda = 0.0001216$ if t is measured in years.

Half Life:- Half life of a radioactive isotope is the time required for half of it to decay. It is denoted by, τ.

In order to find the relationships betn λ and τ, we set, $t=\tau$ and $\frac{N}{N_0} = \frac{1}{2}$ in $N = N_0 e^{-\lambda t}$, we get

$\Rightarrow \frac{1}{2} = e^{-\lambda t} \Rightarrow \tau = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}$, here $\ln 2$ is logarithm of 2 with base e. If the reader wants to determine the formula in terms of logarithm with base 10, then

$$\tau = \frac{\log 2}{\lambda \log e} = \frac{\log 2}{0.4343\lambda}.$$

Q. Carbon extracted from an ancient skull contained only one-sixth as much C-14 as carbon extracted from present day bone. Determine the age of the skull.

Soln: We know that, $\frac{dN}{dt} = -\lambda N \Rightarrow N = N_0 e^{-\lambda t}$.

Put, $\frac{N}{N_0} = \frac{1}{6}$, $\lambda = 0.0001216$, we get

$$\frac{1}{6} = e^{-0.0001216t}$$

$$\Rightarrow \log 1 - \log 6 = -0.0001216t \cdot \log e.$$

$\Rightarrow 0 - 0.778 = -0.0001216 \times 0.4343t$.
$\Rightarrow t = 14731.8$ years.
Therefore, the skull is 14731.8 years old.

Q. A certain moon rock was found to contain equal nos. of potassium and argon atoms. Assume that all the argon is the result of radioactive decay of potassium (its half lief is about 1.28×10^9 years) and that one of every nine potassium atom disintegrates yields an argon atom. What is the age of rock, measured from the time it contained only potassium?

Sol^n: We know that, $\tau = \frac{0.693}{\lambda}$.

$$\Rightarrow \lambda = \frac{0.693}{\tau} = \frac{0.693}{1.28 \times 10^9} = 0.54 \times 10^{-9}.$$

Also, $\frac{dN}{dt} = -\lambda N \Rightarrow N(t) = N_0 e^{-\lambda t}$ —— (i).

At $t=0$, rock contained N_0 atoms of potassium only. Let, $N(t)$ and $A(t)$ be the no. of atoms of potassium and argon at time t. Since the no. of potassium and argon atoms are equal at time t, so. $A(t) = N(t)$.
Also, it is given that one of every nine potassium atom ~~dist~~ disintegrates and yields an argon atom.
Therefore, amount of argon atoms $= A(t) = \frac{1}{9}[N_0 - N(t)]$.
Put, $A(t) = N(t)$, then $N(t) = \frac{1}{9}[N_0 - N(t)]$

$$\Rightarrow N_0 = 10\,N(t) \quad \text{—— (ii)}.$$

using (ii) in (i), we get —

$$N(t) = 10 N(t) \cdot e^{-0.54 \times 10^{-9} t}$$

Solving it, we get $t = 4.26 \times 10^9$ years.
$\therefore$ The age of rock is 4.26×10^9 years.

Drug elimination :-

Let, A_0 be the natural level of a drug in the bloodstream and $A(t)$ is the level of that drug at any time t measured by excess over the natural level of the drug. Then the rate of decrease in the level of $A(t)$ is proportional to the current excess amount. It can be modelled as,

$$\frac{dA}{dt} = -\lambda A \quad , \lambda > 0$$

Also, we have $A = A_0 e^{-\lambda t}$, $\lambda > 0$.

The parameter λ is called the elimination constant of the drug.

Q. When sugar is dissolved in water, the amount A that remains undissolved after t minutes satisfies the differential eq^n $\frac{dA}{dt} = -\lambda A$, $\lambda > 0$. If 25% of the sugar dissolves after 1 min, how long does it take for half of the sugar to dissolve?

Sol^n: Let, A_0 be the amount of sugar dissolved in water initially. At $t=1$, $A = A_0 - 0.25 A_0 = 0.75 A_0$.

We have, $\frac{dA}{dt} = -\lambda A \Rightarrow A = A_0 e^{-\lambda t}$ ——①.

Put, $t=1$ and $A = 0.75 A_0$ in ①, we get $\lambda = 0.28$.

Now, put $A = 0.5 A_0$, $\lambda = 0.28$ in ①, we get

$$0.5 A_0 = A_0 e^{-(0.28)t} \Rightarrow t = 2.47 \text{ min}.$$

It will take 2.47 minutes for half of the sugar to dissolve.

Q. The intansity I of light at a depth of x metres below the ~~f~~ surface of a lake satisfies the d. eqn $\frac{dI}{dx} = -1.4 I$.

(i) At what depth is the intansity half the Intansity I_0 at the surface where $x=0$?

(ii) What is the intansity at a depth of 10m (as a fraction of I_0) ?

(iii) At what depth will the intansity be 1 % of that at the surface ?

Soln : Given that, $\frac{dI}{dx} = -1.4 I$.

$$\Rightarrow \frac{dI}{I} = -1.4\, dx.$$

Integrating, $I = I_0 e^{-1.4x}$ —— (i).

(i) Put $I = 0.5 I_0$ in (i), we get $x = 0.495$ m.

(ii) Put, $x=10$ in (i), we get— $I = I_0 . e^{-1.4 \times 10}$

$$= I_0 e^{-14}$$

$$= 8.32 \times 10^{-7} I_0 .$$

(iii) Put, $I = \frac{1}{100} I_0$ in (i) , we get—

$$\frac{1}{100} I_0 = I_0 e^{-1.4x},$$

Solving we get — $x = 3.29$ m approx.

x