2nd Sem.

5. Newton's Law of Cooling :- Newton's law of cooling, states that rate of change of temperature of a body is proportional to the difference in temperature betn that of surrounding and that of the body itself.

Mathematical Modelling of Newton's Law of Cooling :-

Let, T be the temperature of a body at any time t. Let S be the temperature of surroundings. By rate of change of temperature w.r.t. time t, we mean $\frac{dT}{dt}$.

Then, $\frac{dT}{dt} \propto (T-S)$.

$\Rightarrow \frac{dT}{dt} = -\lambda(T-S)$, where λ is a positive const.

If $T > S \Rightarrow \frac{dT}{dt} < 0$, it means the temperature is a decreasing function of t and the body is cooling. If $T < S$, $\Rightarrow \frac{dT}{dt} > 0$, it means the temperature is an increasing function of t. If we are given the values of λ and S, we should be able to find $T(t)$ which can be helpful in predicting the future temperature of the body.

Q. A pitcher of buttermilk initially at $25^\circ C$ is to be cooled by setting it on front porch, where the temperature is $0^\circ C$. Suppose that the temperature of the buttermilk has dropped to $15^\circ C$ after 20 minutes. When will it be at $5^\circ C$?

<u>Solⁿ</u>: We are given that, at $t=0$, $T=25^\circ C$; at $t=20$, $T=15^\circ C$ and $S=0^\circ C$. We find t when $T=5^\circ C$.

By Newton's law of cooling,

$$\frac{dT}{dt} = -K(T-S) \text{ ; where } \lambda = K.$$

$$\Rightarrow \frac{dT}{T-S} = -K\,dt$$

Integrating, $\log(T-S) = -Kt + \log c$; where c is const.

$$\Rightarrow T-S = c e^{-Kt} \quad \text{—— (1).}$$

Putting, $t=0$, $T=25$ & $S=0$ in (1), we get $c=25$.

Now, put, $t=20$, $T=15$ and $c=25$ in (1), we get

$$15-0 = 25 e^{-K(20)} \Rightarrow 0.6 = e^{-K(20)}$$

$$\Rightarrow \log 0.6 = -20K \log e \Rightarrow -0.2218 = -20K(0.4343)$$

$$\Rightarrow K = 0.025.$$

Putting, $K=0.025$, $T=5$, $c=25$ & $S=0$ in (1), we get

$$5 = 25 e^{-0.025t}$$

$$\Rightarrow \log 5 = \log 25 - 0.025t \log e.$$

$$\Rightarrow 0.699 = 1.3979 - 0.025t \cdot (0.4343)$$

$$\Rightarrow t = 64.37 \text{ min}$$

6. Growth and Decay :-

Natural Growth Equation :- The d. eqⁿ $\frac{dx}{dt} = \mu x$, $x(t) > 0$, $\mu > 0$ is called a natural growth eqⁿ or exponential eqⁿ.

Natural Decay eqⁿ :- The d. eqⁿ $\frac{dx}{dt} = \mu x$; $x(t) > 0, \mu < 0$ is called a natural decay eqⁿ.

Population growth : Let $P(t)$ be the population having constant birth and death rates. Then the time rate of change of population $P(t)$ is proportional to the size of the population. Then we have—

$\frac{dP}{dt} = \lambda P$, where λ is a constant of proportionality.

$\Rightarrow \frac{dP}{P} = \lambda dt$.

Integrating, $\log P = \lambda t + c$ —— ①.

Let the population be P_0 initially. It means $P(0) = P_0$

i.e. $P = P_0$ at $t = 0$. So, ① $\Rightarrow \log P_0 = c$.

Thus, ① $\Rightarrow \log P = \lambda t + \log P_0 \Rightarrow P = P_0 e^{\lambda t}$.

This is the population at any time t if the initial population is P_0.

Q. Suppose a population P of rodents, and their number is increasing at the rate of $\frac{dP}{dt} = 1$ rodent per month when there are $P = 10$ rodents. How long will it take for this population to grow to a hundred rodents? To a thousand? What's happening here?

Solⁿ: We have, $\frac{dP}{dt} = \lambda P^2$. Put, $\frac{dP}{dt} = 1$ in it, we get

$$1 = \lambda (10)^2 \Rightarrow \lambda = \frac{1}{100}$$

Then, $\frac{dP}{dt} = \frac{1}{100} P^2 \Rightarrow \frac{dP}{P^2} = \frac{1}{100} dt$

Integrating, $-\frac{1}{P} = \frac{1}{100}t + c$. —— ①.

Put, $P(0) = 2$, i.e. $P = 2$ when $t = 0$.

$\therefore$ ① $\Rightarrow$ $c = -\frac{1}{2}$.

Thus, ① becomes, $-\frac{1}{P} = \frac{1}{100}t + -\frac{1}{2}$

If $P = 100$ then, $-\frac{1}{100} = \frac{1}{100}t - \frac{1}{2}$

$\Rightarrow t = 4.9$ months.

If $P = 1000$ then, $-\frac{1}{1000} = \frac{1}{100}t - \frac{1}{2}$.

$\Rightarrow t = 49.9$ months.

Thus, it appears that $P(t)$ grows without bound as t approaches 50.

Q. A certain city had a population of 25000 in 1960 and a population of 30000 in 1970. Assume that its population will continue to grow exponentially at a constant rate. What population can its city planners expect in the year 2000.

Soln: Let the year 1960 be taken as origin. Then, at $t = 0$, $P = 25000$ i.e. $P_0 = 25000$ at $t = 10$, $P = 30000$.

We have, $\frac{dP}{dt} = \lambda P \Rightarrow \frac{dP}{P} = \lambda dt$.

Integrating, $P = P_0 e^{\lambda t}$.

$\therefore$ $P = 25000 e^{\lambda t}$.

Now, put $t = 10$ and $P = 30000$, we get

$$30000 = 25000\, e^{\lambda(10)}$$

$$\Rightarrow e^{10\lambda} = 1 \cdot 2 \Rightarrow 10\lambda \log e = \log 1 \cdot 2.$$

$$\Rightarrow 10\lambda = \frac{0 \cdot 079}{0 \cdot 4343} \Rightarrow \lambda = 0 \cdot 018.$$

$$\therefore \quad P = 25000\, e^{0 \cdot 018 t}$$

For the year. 2000, $t = 2000 - 1960 = 40$.

Put $t = 40$ in $P = 25000 . e^{0 \cdot 018 t}$, we get

$$P = 25000 . e^{0 \cdot 018 \times 40} = 25000 \times 2 \cdot 054 = 51350 \text{ person}$$