3. Swimmer's Problem :-

Consider a northward flowing river of width $w = 2a$. The lines $x = \pm a$ represents the banks of the river and y axis its centre. If the swimmer starts at the point $(-a, 0)$ on the west and swims towards east with constant speed v_s. Then the diff. eqⁿ,

$$\frac{dy}{dx} = \frac{v_0}{v_s}\left(1 - \frac{x^2}{a^2}\right)$$ gives the swimmer's trajectory $y = y(x)$ as he crosses the river.

Here, v_0 is the midstream velocity, x is the distance from the centre of the river, v_s is the velocity of the swimmer.

Q. Suppose a 1 mile wide river has 9 miles/h midstream velocity. A swimmer is swimming with a velocity of 3 miles/h. Find how far downstream the swimmer drifts as he crosses the river.

Solⁿ: The width of the river $= 2a = 1$.

Therefore, $a = \frac{1}{2}$. Also $v_0 = 9$, $v_s = 3$.

Put these values in $\frac{dy}{dx} = \frac{v_0}{v_s}\left(1 - \frac{x^2}{a^2}\right)$, we get—

$$\frac{dy}{dx} = \frac{9}{3}\left(1 - \frac{x^2}{(1/2)^2}\right) = 3(1 - 4x^2).$$

$$\Rightarrow dy = 3(1 - 4x^2)\,dx$$

Integrating, $y(x) = 3x - 4x^3 + c$ —— ①.

Since the swimmer starts at $x = -\frac{1}{2}$, therefore, $y(-\frac{1}{2}) = 0$ $\therefore$ ① $\Rightarrow$ $c = 1$.

Thus, ① $\Rightarrow$ $y(x) = 3x - 4x^3 + 1$.

Then, $x\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right)^3 + 1 = 2.$

So, the swimmer drifts 2 miles downstream while he swims 1 mile across the river.

4. **Torricelli's Law :-** Torricelli's law states that time rate of change of the volume of the water in a draining tank is proportional to the square root of the depth of water in the tank.

· **Mathematical Modelling of Torricelli's Law :**

Consider a water tank with a bottom hole of area a from which the water is leaking. Let, $y(t)$ denotes the depth of water at time t. Let $V(t)$ be the volume of the water in the draining tank. Under ideal conditions, the velocity of water draining through the hole is $v = \sqrt{2gy}$, where, g is the acceleration due to gravity. It is actually the velocity of a water drop when it falls freely from the surface of water to the hole.

Then, $\frac{dV}{dt} = -av = -a\sqrt{2gy}$ —— (i).

Let, $\lambda = a\sqrt{2g}$ then $\frac{dV}{d\lambda} = -\lambda\sqrt{y}$ —— (ii).

Let, $A(y)$ be the horizontal cross-sectional area of the water tank at a height y above the hole.

Then, $V = \int_0^y A(y)\, dy.$

By fundamental theorem of calculus, $\frac{dV}{dy} = A(y).$

$\therefore \frac{dV}{dt} = \frac{dV}{dy} \cdot \frac{dy}{dt} = A(y)\frac{dy}{dt}$ —— (iii).

From (i), (ii) and (iii), we get—

$$A(y)\,\frac{dy}{dt} = -a\sqrt{2gy} = -\lambda\sqrt{y}.$$

Q. (i) A tank is shaped like a vertical cylinder; it initially contains water to a depth of 9 ft and a bottom plug is removed at time $t=0$ hours. After 1 hour, the depth of the water has dropped to 4 ft. How long does it take for all the water to drain from the tank?

(ii) Suppose, that the tank has a radius of 3 ft and its bottom hole is circular with radius 1 inch. How long will it take the water (initially 9 ft deep) to drain completely?

Sol^n: (i) According to Torricelli's law, if $y(t)$ denotes the depth of water in the tank at time t, then

$$A(y)\,\frac{dy}{dt} = -\lambda\sqrt{y}.$$

where $A(y)$ denotes the horizontal cross-sectional area of tank at height y above the plug.

$$\frac{dy}{dt} = -\frac{\lambda}{A(y)}\sqrt{y}.$$

$$\Rightarrow \frac{dy}{dt} = -h\sqrt{y}, \text{ where } h = \frac{\lambda}{A(y)}$$

since $A(y)$ is constant.

$$\Rightarrow \frac{dy}{\sqrt{y}} = -h\,dt$$

Integrating, $2\sqrt{y} = -ht + c$, where c is the constant of integration.

Put, $y(0) = 9$ i.e. $y=9$ at $t=0$, we get $-$ $c=6$.

then, $2\sqrt{y} = -ht + 6$.

Put, $y(1) = 4$ i.e. $y=4$ at $t=1$, we get $-$ $h=2$,

then $2\sqrt{y} = -2t + 6$.

In order to determine t when all the water get drain from the tank, we put $y=0$ and find t.

This gives, $t = 3$. Therefore the tank gets empty in 3 hours.

(ii) Let, a be the area of the hole of radius 1 inch i.e. $1/12$ ft.

Then, $a = \pi\left(\frac{1}{12}\right)^2$

Also, $A(y) = \pi(3)^2 = 9\pi$.

We know that, $A(y)\frac{dy}{dt} = -a\sqrt{2gy}$, where $g = 32\,ft/s^2$

$\Rightarrow 9\pi \frac{dy}{dt} = -\pi\left(\frac{1}{12}\right)^2\sqrt{2 \times 32 \times y}$.

$\Rightarrow 162\frac{dy}{dt} = -\sqrt{y}$.

$\Rightarrow 162\frac{dy}{\sqrt{y}} = -dt$

Integrating, $324\sqrt{y} = -t + c$.

Put, $t=0$ and $y=9$, we get $c = 972$.

$\therefore 324\sqrt{y} = -t + 972$.

Now, we find t, when $y=0$. Put $y=0$, in the above expression we get, $t = 972$.

Therefore, the tank will take 972 seconds to drain completely.

Q. Suppose that a cylindrical tank initially containing V_0 gallons of water drains through a bottom hole in T minutes. Use Torricelli's law to show that the volume of water in the tank after $t \le T$ minutes is,

$$V = V_0 \left[1 - \frac{t}{T}\right]^2.$$

Sol^n: According to Torricelli's law, $\frac{dy}{dt} = -\lambda\sqrt{y}$.

$$\Rightarrow \frac{dy}{\sqrt{y}} = -\lambda dt$$

Integrating, $2\sqrt{y} = -\lambda t + c$. —— (i).

Put, $y(0) = h$, we get $c = 2\sqrt{h}$.

$\therefore$ (i) $\Rightarrow$ $2\sqrt{y} = -\lambda t + 2\sqrt{h}$ —— (ii).

At, $t = T$, $y = 0$,

$\therefore$ (ii) $\Rightarrow$ $2\sqrt{y} = -\lambda t + 2\sqrt{h}$

Put, this value of λ in (ii), we get

$$2\sqrt{y} = \frac{2\sqrt{h}}{T} + 2\sqrt{h}$$

$$\Rightarrow \sqrt{y} = \sqrt{h}\left[1 - \frac{t}{T}\right]$$

$$\Rightarrow y = h\left[1 - \frac{t}{T}\right]^2.$$

If r denotes the radius of cylinder, then

$$V(y) = \pi r^2 y = \pi r^2 h\left[1 - \frac{t}{T}\right]^2$$

$$= V_0\left[1 - \frac{t}{T}\right]^2$$