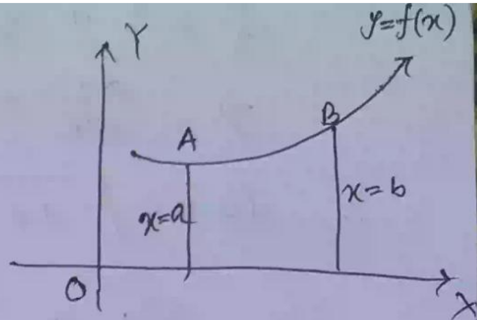
Centre of Gravity of an arc :→



The centre of gravity of the arc $y=f(x)$ extending from $x=a$ to $x=b$ is given by,

$$\bar{x} = \frac{\int x\,dm}{\int dm} = \frac{\int_{x=a}^{b} x\rho\, ds}{\int_{x=a}^{b} \rho\, ds} = \frac{\int_{x=a}^{b} x\, ds}{\int_{x=a}^{b} ds}$$

and $$\bar{y} = \frac{\int y\,dm}{\int dm} = \frac{\int_{x=a}^{b} \rho y\, ds}{\int_{x=a}^{b} \rho\, ds} = \frac{\int_{x=a}^{b} y\, ds}{\int_{x=a}^{b} ds}.$$

where, ρ is the density of the arc and ds is an elementary arc.

1.Q. Find the C.G. of an arc of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ lying in the first quadrant. [2005, 06, 08, 10, 2015]

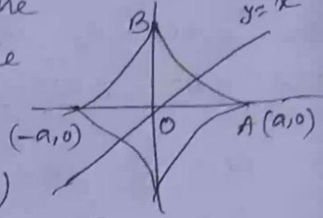
Sol^n: The given eq^n is $x^{2/3}+y^{2/3}=a^{2/3}$ —— ①.

Putting $y=0$ in ①, we get

$$x^{2/3}=a^{2/3} \Rightarrow x = \pm a.$$

i.e. the curve meet the x axis $(a,0)$ and $(-a,0)$.

The curve ① is symmetrical about the line $y=x$ and its C.G. must lie on this line.



For the first quadrant, let $(\bar{x},\bar{y})$ be the req^d C.G. where x varies from 0 to a.

Then,
$$\bar{x} = \frac{\int_{x=0}^{a} x\,ds}{\int_{x=0}^{a} ds} \quad \text{and} \quad \bar{y} = \frac{\int_{x=0}^{a} y\,ds}{\int_{x=0}^{a} ds}.$$

Now, diff. ① w.r.t. x, we have—

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}} \cdot \frac{dy}{dx} = 0.$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}.$$

$$\therefore ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\frac{y^{2/3}}{x^{2/3}}}\,dx = \sqrt{\frac{x^{2/3}+y^{2/3}}{x^{2/3}}}\,dx$$

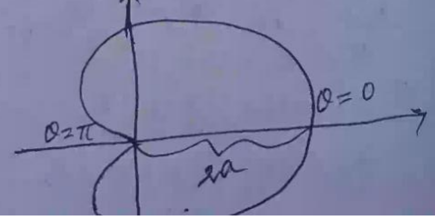
$$= \sqrt{\frac{a^{2/3}}{x^{2/3}}}\,dx = \frac{a^{1/3}}{x^{1/3}}\,dx.$$

$$\therefore \bar{x} = \frac{\int_{x=0}^{a} x \cdot \frac{a^{1/3}}{x^{1/3}}\,dx}{\int_{x=0}^{a} \frac{a^{1/3}}{x^{1/3}}\,dx} = \frac{2a}{5}.$$

Since, the arc is symmetrical about the line $y=x$, so we have, $\bar{x}=\bar{y}$.

Hence, the req^d C.G. is $\left(\frac{2a}{5}, \frac{2a}{5}\right)$.

2.Q. Find the C.G. of an arc $r=a(1+\cos\theta)$ lying above the initial line.



Sol^n: The given curve is,

$$r = a(1+\cos\theta) \quad \text{—— ①}.$$

$\therefore \dfrac{dr}{d\theta} = -a\sin\theta$

and $\dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2} = \sqrt{a^2\{1+2\cos\theta+\cos^2\theta+\sin^2\theta\}}$

$= 2a\cos\theta/2$ —— (ii).

The given curve is symmetrical about the initial line.

Let, $(\bar{x},\bar{y})$ be the co-ordinate of the req^d^ c.G.

$$\therefore \bar{x} = \frac{\int x\,ds}{\int ds} = \frac{\int_{\theta=0}^{\pi} r\cos\theta \cdot 2a\cos\theta/2\,d\theta}{\int_{\theta=0}^{\pi} 2a\cos\theta/2\,d\theta}$$

$$= \frac{\int_0^{\pi} a(1+\cos\theta)\cos\theta\cdot\cos\theta/2\,d\theta}{\int_0^{\pi}\cos\theta/2\,d\theta} = \frac{4a}{5}.$$

and,
$$\bar{y} = \frac{\int y\,ds}{\int ds} = \frac{\int_{\theta=0}^{\pi} r\sin\theta\cdot 2a\cos\theta/2\,d\theta}{\int_{\theta=0}^{\pi} 2a\cos\theta/2\,d\theta}$$

$$= \frac{\int_0^{\pi} a(1+\cos\theta)\sin\theta\cdot\cos\theta/2\,d\theta}{\int_0^{\pi}\cos\theta/2\,d\theta} = \cancel{\frac{8a}{5}}\ \frac{4a}{5}$$

Therefore, the required c.G. is $\left(\frac{4a}{5}, \frac{4a}{5}\right)$