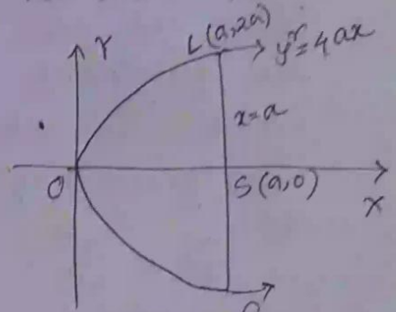
Problems

1.Q. Find the C.G. of the area bounded by the parabola $y^2 = 4ax$, the axis of x and the latusrectum. [2014, 2018]



Sol^n: The eq^n of the parabola is $y^2 = 4ax$ —— (1).

Then the co-ordinates of one end of the latusrectum is $L(a, 2a)$. We are to find the C.G. of the area OSL. For this, we consider an elementary strip of the area OSL parallel to y-axis. Its area is $y\,dx$ and its C.G. is the point $\left(x, \frac{y}{2}\right)$.

If $(\bar{x}, \bar{y})$ be the C.G. of the area OSL where x lies from '0' to 'a', we have—

$$\bar{x} = \frac{\int_{x=0}^{a} xy\,dx}{\int_{x=0}^{a} y\,dx} \quad \text{and} \quad \bar{y} = \frac{1}{2}\frac{\int_{x=0}^{a} y^2\,dx}{\int_{x=0}^{a} y\,dx}$$

Now,

$$I_1 = \int_{x=0}^{a} xy\,dx = \int_0^a x \cdot 2\sqrt{ax}\,dx = \frac{4}{5}a^3.$$

$$I_2 = \int_{x=0}^{a} y^2\,dx = \int_0^a 4ax\,dx = 2a^3.$$

$$I_3 = \int_{x=0}^{a} y\,dx = \int_0^a 2\sqrt{4x}\,dx = \frac{4}{3}a^2.$$

Therefore,

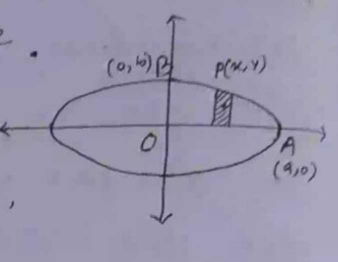
$$\bar{x} = \frac{I_1}{I_3} = \frac{4/5\,a^3}{4/3\,a^2} = \frac{3}{5}a.$$

$$\bar{y} = \frac{1}{2}\frac{I_2}{I_3} = \frac{1}{2}\frac{2a^3}{\frac{4}{3}a^2} = \frac{3}{4}a.$$

Thus, the req^d C.G. is $\left(\frac{3}{5}a, \frac{3}{4}a\right)$

2.Q. Determine the c.G. of a quadrant of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol^n: Let us suppose that OAB be the quadrant of the ellipse of which the c.G. is to be evaluated.



The given eq^n of the ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{—— (i)}.$$

The parametric eq^n of the ellipse (i) are $x = a\cos t$ and $y = b\sin t$.

All the point $A(a,0)$, $t=0$ and at $B(0,b)$, we have $t = \frac{\pi}{2}$.

Let, $(\bar{x}, \bar{y})$ be the req^d c.G. of the area OAB. We take an elementary strip at $P(x,y)$ of thickness δx, then the area of the strip is $y\delta x$ and its c.G. can be taken at the point of $(x, y/2)$. Then we have—

$$\bar{x} = \frac{\int_{x=0}^{a} xy\,dx}{\int_{x=0}^{a} y\,dx} = \frac{\int_{t=\frac{\pi}{2}}^{0} a\cos t \cdot b\sin t \cdot (-a\sin t\,dt)}{\int_{t=\frac{\pi}{2}}^{0} b\sin t \cdot (-a\sin t \cdot dt)}$$

$$= \frac{a^2 b \int_0^{\pi/2} \sin^2 t \cdot \cos t \cdot dt}{ab \int_0^{\pi/2} \sin^2 t\, dt} = \frac{4a}{3\pi}$$

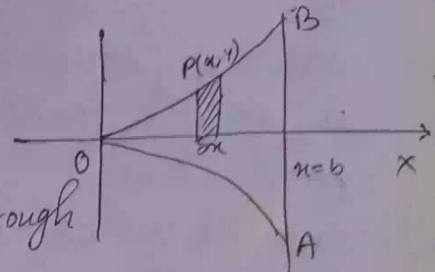
and
$$\bar{y} = \frac{1}{2} \cdot \frac{\int_{x=0}^{a} y^2\,dx}{\int_{x=0}^{a} y\,dx} = \frac{1}{2} \frac{\int_{t=\frac{\pi}{2}}^{0} b^2\sin^2 t \cdot (-a\sin t)\,dt}{\int_{t=\frac{\pi}{2}}^{0} b\sin t\,(-a\sin t)\,dt}$$

$$= \frac{4b}{3\pi}.$$

Therefore, $\left(\frac{4a}{3\pi}, \frac{4b}{3\pi}\right)$ is the req^d c.G.

3.Q. Find the position of the centroid of the area of the curve $ay^2 = x^3$ bet^n the origin and $x = 6$.

Sol^n: Given eq^n of the curve is



$$ay^2 = x^3 \text{ ——— } (1).$$

The curve is symmetrical about the axis x and it passes through the origin.

Let us consider an elementary strip at $P(x, y)$ with thickness δx.

Also, let, $(\bar{x}, \bar{y})$ be the co-ordinate of C.G. at the area of OAB, which is included ~~bet^n~~ by the curve (1) and the line $x = b$; where x lies bet^n $x = 0$ and $x = b$.

Also, from symmetric, we have $\bar{y} = 0$.

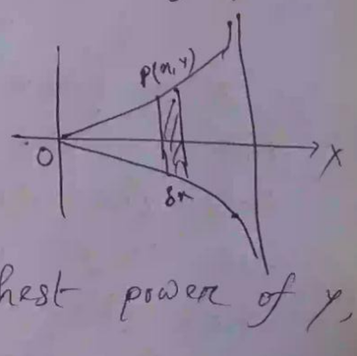
$$\therefore \quad \bar{x} = \frac{\int_{x=0}^{b} xy\, dx}{\int_{x=0}^{b} y\, dx} = \frac{\int_0^b x \cdot \frac{x^{3/2}}{\sqrt{a}}\, dx}{\int_0^b \frac{x^{3/2}}{\sqrt{a}}\, dx} \text{ ; by (1)}$$

$$= \frac{5b}{7}.$$

Therefore, the req^d C.G. is $\left(\frac{5b}{7}, 0\right)$ //

4.Q. Find the C.G. of the area included bet^n the curve $y^2(2a - x) = x^3$ and its asymptots.

Sol^n: The given curve is,



$$y^2(2a - x) = x^3 \text{ ——— } (1).$$

It is symmetrical about the x-axis.

. It is passes through the origin.

Now, equating the coefficient of highest power of y, we get,

$$2a - x = 0$$

$$\Rightarrow x = 2a \text{ ——— } (11).$$

We are reqd to find the c.G. of the area included betn the curve (I) and its asymptotes (II).

Now, we consider an elementory strip of thickness is at the point $P(x,y)$ its c.G. will be at the point $(x,0)$.

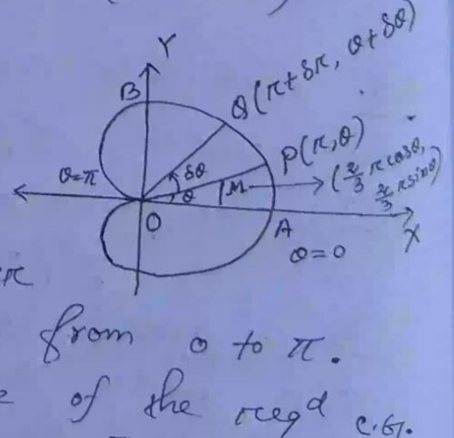
Let, $(\bar{x},\bar{y})$ be the required c.G. then from symmetry, $\bar{y}=0$ and,

$$\bar{x} = \frac{\int_{x=0}^{2a} xy\,dx}{\int_{x=0}^{2a} y\,dx} = \frac{\int_{0}^{2a} \frac{x \cdot x^{3/2}}{\sqrt{2a-x}}\,dx}{\int_{x=0}^{2a} \frac{x^{3/2}}{\sqrt{2a-x}}\,dx} = \frac{5a}{3}$$

$\therefore$ Reqd c.G. is, $\left(\frac{5a}{3}; 0\right)$.

5.Q Find the position of c.G. of the area of the curve $x^{2/3}+y^{2/3}=a^{2/3}$ lying in the positive quadrant. [2015]

6.Q. Find the position of c.G. of the cardioid of the area whose eqn is $r=a(1+\cos\theta)$.
[2015, 2017]



Soln: The given eqn is,

$$r = a(1+\cos\theta) \quad \text{—(I)}.$$

It is symmetrical about the initial line. And for the upper half for the curve θ varies from 0 to π.
Let, $(\bar{x},\bar{y})$ be the co-ordinate of the reqd c.G.
Then from symmetry, we have— $\bar{y}=0$.
Also, by symmetry the x-coordinate of c.G. of the cardioid is same as the x-coordinate of the

C.G. of the upper half ABO of this area.

Let us consider an elementary strip POQ of ABO.

The area of the strip is $\frac{1}{2} r^2 \delta\theta$.

And the x-co-ordinate of C.G. can be taken as, $\frac{2}{3} r\cos\theta$.

$$\therefore \quad \bar{x} = \frac{2}{3} \frac{\int_{\theta=0}^{\pi} r^3 \cos\theta \, d\theta}{\int_{\theta=0}^{\pi} r^2 \, d\theta} = \frac{2}{3} \frac{\int_0^{\pi} a^3(1+\cos\theta)^3 \cos\theta \cdot d\theta}{\int_0^{\pi} a^2(1+\cos\theta)^2 \, d\theta}$$

$$= \frac{2a}{3.} \frac{\int_0^{\pi} (1+\cos\theta)^3 \cos\theta \, d\theta}{\int_0^{\pi} (1+\cos\theta)^2 \, d\theta} = \; = \; = \frac{5}{6} a .$$

$\therefore$ Req^d C.G. is $\left(\frac{5a}{6}, 0\right)$.//

$y=f(x)$