Orthogonal Trajectory :-

Trajectory: A trajectory of the given system of curves is defined as a curve which cuts every member of a family of curves according to given law.

Orthogonal Trajectory: If the curve cuts every member of a given family of the curves at right angles then curve is known as orthogonal trajectory of the family.

Differential Eqns of orth. trajectories :-

1. **Cartesian Co-ordinates Form** :- Consider a given family of curves be represented by $f(x,y,c)=0$ —— (i).

where, c is an arbitrary constant/parameter.

Diff. (i) w.r.t. x and eliminating c with the help of (i), we get-

$$F\left(x,y,\frac{dy}{dx}\right)=0 \quad \text{—— (ii).}$$

which is the d. eqn of the given family and $\frac{dy}{dx}$ is gradient of tangent at (x,y).

Let, (X,Y) be the tangent co-ordinates of any point of an orthogonal trajectory of (i), so that

$\frac{dY}{dX}$ = gradient tangent to the orth. tray.

Therefore, at point of intersection of (i) and its orthogonal trajectory,

$$\frac{dy}{dx}\cdot\frac{dY}{dX}=-1.$$

$$\Rightarrow \frac{dY}{dX}=-\frac{dx}{dy} \quad \text{—— (iii).}$$

at the point of intersection $X=x,\ Y=y$ —— (iv).

Now, replacing x by X, y by Y and $\frac{dy}{dx}$ by $-\frac{dX}{dY}$ we have,

$$F\left(X, Y, -\frac{dx}{dy}\right) = 0.$$

This is the d. eq^n of orth. trajectories. When integrating this eq^n we get the eq^n of family of orthogonal trajectory of (i).

2. Polar Co-ordinates:- Let the eq^n of given family of the curve be, $F(r, \theta, c) = 0$ —— (i).

where, c is a parameter.

Diff. (i) w.r.t. 'θ' and eliminating c, we get—

$$F\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \text{ —— (ii).}$$

This is the d. eq^n of given family.

But, $\tan\phi_1 = \frac{r\,d\theta}{dr}$ for the given family (i).

ϕ_1 be the angle bet^n tangent and radius vector at (r, θ).

If (R, H) be the current co-ordinates of any point on orthogonal trajectory and ϕ_2 angle bet^n radius and tangent,

$$\therefore \tan\phi_2 = R\frac{dH}{dR}.$$

If it intersect at right angles,

$$\therefore \tan\phi_1 \cdot \tan\phi_2 = -1.$$

$$\Rightarrow r\frac{d\theta}{dr} \cdot R\frac{dH}{dR} = -1,$$

$$\Rightarrow r\frac{d\theta}{dr} = -\frac{1}{R}\frac{dR}{dH}.$$

at point of intersection $R = r$, $H = \theta$.

$\therefore$ Replacing $\frac{r\,d\theta}{dr}$ by $-\frac{1}{r}\frac{dr}{d\theta}$ in (ii), we get— diff. eq^n of orthogonal trajectory (i).

Q.1 Find the orthogonal trajectories of family of the curves $xy = k^2$.

Sol^n: Given, curve —

$$xy = k^2 \quad \text{——} (i).$$

k is parameter.

Diff. (i) w.r.t. x, we get —

$$y + x \cdot \frac{dy}{dx} = 0 \quad \text{——} (ii).$$

Now, replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get—

$$y + x \cdot \left(-\frac{dx}{dy}\right) = 0$$

$$\Rightarrow y - x\frac{dx}{dy} = 0 \quad \text{——} (iii)$$

which is variable separable form d. eq^n.

$\therefore$ (iii) $\Rightarrow$ $y\,dy - x\,dx = 0$

Integrating, $y^2 - x^2 = c^2$.

which represent the orthogonal trajectory of (i).

Q.2. Find the orthogonal trajectory of the family of cardioides, $r = a(1+\cos\theta)$.

Sol^n: Given, $r = a(1+\cos\theta)$ —— (i)

where, a is parameter.

Diff. (i) w.r.t. 'θ', we get —

$$\frac{dr}{d\theta} = -a\sin\theta. \quad \text{——} (ii).$$

From (i) and (ii), we get —

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-a\sin\theta}{a(1+\cos\theta)} = -\frac{2\sin\theta/2 \cdot \cos\theta/2}{2\cos^2\theta/2} = -\tan\frac{\theta}{2}.$$

$$\Rightarrow \frac{r\,d\theta}{dr} = -\cot\frac{\theta}{2} \quad \text{——} (iii).$$

Now, replacing $\frac{r\,d\theta}{dr}$ by $-\frac{1}{r}\frac{dr}{d\theta}$, we get—

$$-\frac{1}{r}\frac{dr}{d\theta} = -\cot\frac{\theta}{2}$$

$$\Rightarrow -\frac{1}{r}dr = \cot\frac{\theta}{2} \cdot d\theta \quad \text{——} (iv).$$

Integrating, $\log r = 2\log \sin\frac{\theta}{2} + \log a$.

$$\Rightarrow r = a(1-\cos\theta).$$

which is the req^d eq^n of orth trajectory.

Q.3 Find the orthogonal trajectory of the family of the curve, $y^2 = 4a(x+a)$ belong to ~~itself~~

Sol^n: The given eq^n of the family of the curve is,

$$y^2 = 4a(x+a) \text{ --- (i).}$$

where, a is parameter.

Diff. (i) w.r.t x, we get—

$$2y\frac{dy}{dx} = 4a \Rightarrow y\frac{dy}{dx} = 2a \text{ --- (ii).}$$

$\Rightarrow \cancel{y^2}$

eliminating 'a' bet^n (i) and (ii), we get—

$$y^2 = 2y\frac{dy}{dx}\left(x + \frac{1}{2}y\frac{dy}{dx}\right).$$

$$\Rightarrow y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2 \text{ --- (iii).}$$

Now, replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get—

$$y^2 = 2xy\left(-\frac{dx}{dy}\right) + y^2\left(-\frac{dx}{dy}\right)^2.$$

$$\Rightarrow y^2 = -2xy\frac{dx}{dy} + y^2\left(\frac{dx}{dy}\right)^2.$$

$$\Rightarrow y^2 = \left(\frac{dx}{dy}\right)^2\left[-2xy\cdot\frac{dy}{dx} + y^2\right]$$

$$\Rightarrow y^2\left(\frac{dy}{dx}\right)^2 = -2xy\frac{dy}{dx} + y^2.$$

$$\Rightarrow y^2 = -2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2 \text{ --- (iv).}$$

comparing (iii) & (iv), we see that they being identical the family (i) is self orthogonal.

Home Works

Find the orthogonal trajectories of the following family of curves :

1. $y = ax^2$
2. $x^2 + y^2 + kx = 0$.
3. $x^2 + y^2 = a^2$
4. $r = e^{a\theta}$.
5. $r = a + \sin 5\theta$.
6. $r = a(1 - \cos\theta)$
7. $\left(\frac{dy}{dx}\right)^2 = \frac{y}{x}$.
8. $r = c(\cos\theta + 6\sin\theta)$.
9. $x^{2/3} + y^{2/3} = a^{2/3}$.