Maxwell's Equations; Electromagnetic Waves

In 1845, Faraday demonstrated that a magnetic field produces a measurable effect on a beam of light. This encouraged him to guess that light involves oscillation of electric and magnetic field lines, but he could not prove this mathematically. James Clerk Maxwell, a young admirer of Faraday, believed that the closeness of these two numbers, speed of light and the inverse square root of ε_0 and μ_0, was more than just coincidence and decide to develop Faraday's hypothesis. In 1865, he predicted the existence of electromagnetic waves that propagate at the speed of light. He proposed the four laws on the basis of basic laws of electrostatic and magnetism. First three laws are general equations and fourth one was derived from steady state observation of current and variation of electric displacement with time.

Differential Form of Maxwell's equations

1. $\nabla\cdot D=\rho$, $\nabla\cdot E=\rho/\epsilon_0$ — Gauss's law of electrostatics
2. $\nabla\cdot B=0$ — Gauss's law for magnetism
3. $\nabla\times E=-\partial B/\partial t$ — Faraday's law of induction
4. $\nabla\times H= J+\partial D/\partial t$, $\nabla\times B=\mu_0 J+\mu_0\epsilon_0\partial E/\partial t$ — Ampère's law

$D=\epsilon_0 E$, $B=\mu_0 H$ applicable in free space.
$D=\epsilon E$, $B=\mu H$ applicable in any other medium.

D= electric flux density/displacement field (Unit: As/m2)
E= electric field intensity (Unit: V/m)
ρ= electric charge density (As/m3)
H= magnetic field intensity (Unit: A/m)
B= magnetic flux density (Unit: Tesla=Vs/m2)
J= electric current density (A/m2)
ϵ_0=permittivity of free space
μ_0=permeability of free space
ϵ= absolute permittivity of any medium
μ= absolute permeability of any medium

Integral form of Maxwell's low

1. $\iint D\cdot dS=\iiint \rho dV$, Applying Gauss low $\iiint(\nabla\cdot F)dV=\iint F\cdot d\,S$ in differential form of Maxwell's equation

2. $\iint B\cdot dS=0$, Applying Gauss low $\iiint(\nabla\cdot F)dV=\iint F\cdot d\,S$ in differential form of Maxwell's equation

3. $\int E\cdot d\,l=-\iint \partial B/\partial t\cdot dS$, Applying stokes law $\iint(\nabla\times F)\cdot d\,S=\int F\cdot d\,l$ in differential form of Maxwell's

4. $\int H\cdot d\,l=\iint J\cdot d\,S+\iint\partial D/\partial t\cdot dS$ Applying stokes law , $\iint(\nabla\times F)\cdot d\,S=\int F\cdot d\,l$ in differential form of Maxwell's

Derivation of the Wave Equation in free space.

Mathematical operation of Faraday's law and Ampere-Maxwell law leads directly to a wave equation for the electric and magnetic field.

The Maxwell's equations are written as

$\nabla \cdot D=\rho$, $\nabla \cdot E=\rho/\epsilon_0$	Gauss's law of electrostatics
$\nabla \cdot B=0$	Gauss's law for magnetism
$\nabla \times E=-\partial B/\partial t$	Faraday's law of induction
$\nabla \times H= J+\partial D/\partial t$, $\nabla \times B=\mu_0 J+\mu_0\epsilon_0\partial E/\partial t$	Ampère's law

In case of non-charged, current free homogeneous, linear, and isotropic medium, $\rho=0$, J=0 and Maxwell's equations can be written as

$\nabla \cdot E=0$ …………………..1

$\nabla \cdot B=0$ ……………………2

$\nabla \times E=-\partial B/\partial t$ ………………3

$\nabla \times B=\mu_0\epsilon_0\partial E/\partial t$………………4

Taking curl of equation (3) we get

$\nabla \times(\nabla \times E) =-\partial (\nabla \times B)/\partial t$ {curl of curl of a vector; $\nabla \times\nabla \times A=\nabla(\nabla \cdot A)-\nabla^2 A$}

Or $\nabla (\nabla \cdot E) -\nabla^2 E=-\partial(\mu_0\epsilon_0\partial E/\partial t)/ \partial t$ {using equation 4}

Or $\nabla^2 E= \mu_0\epsilon_0\partial^2 E/\partial t^2$…………………..5 {using equation 1}

This is the wave equation for Electric Field

Again, we can find out wave equation for Magnetic Field

Let us take curl of equation (4) we get

$\nabla \times\nabla \times B=\mu_0\epsilon_0\partial(\nabla \times E)/\partial t$ { $\nabla \times\nabla \times A=\nabla(\nabla \cdot A)-\nabla^2 A$}

$\nabla(\nabla \cdot B) -\nabla^2 B=-\mu_0\epsilon_0\partial^2 B/\partial t^2$ {using equation 3}

$\nabla^2 B=\mu_0\epsilon_0\partial^2 B/\partial t^2$ ……………………..6 {using equation 2}

This is the wave equation for Magnetic Field

The wave equation propagating in x direction can be written as $\partial^2 y/\partial x^2 = 1/v^2 (\partial^2 y/\partial t^2)$ in Free space and plane wave solution of this equation can be written as $A(r, t)= A_0 \cdot e^{i(\omega t-k\cdot x)}$ where k is a wave vector pointing in direction of wave propagation. Wave is unconfined in plane orthogonal to wave direction, i.e. has surfaces of constant phase (wave fronts), wave vector k is perpendicular to the wave front. From the solution of wave equation mathematically we can find out

$\nabla A= A_0\, e^{\, i(\omega t-k\cdot r)} \times (-ik)$

Or $\nabla^2 A= A_0\, e^{\, i(\omega t-k\cdot r)} \times (-ik)(-ik)$

So $\nabla^2 \rightarrow - k^2$………………………7

Again,

$\partial^2 A/\partial t^2 = A_0\, e^{\, i(\omega t-k\cdot r)} \times (i\omega) \times (i\omega)$

So $\partial^2/\partial t^2 \rightarrow -\omega^2$…………………..8

Putting the value of ∇^2 and $\partial^2/\partial t^2$ from the equation 7 and 8 in the wave equation for electric field or magnetic field (equation 5 and 6) we get,

$\nabla^2 E= \mu_0\epsilon_0\partial^2 E/\partial t^2$

Or $-k^2 E = -\omega^2 \mu_0 \epsilon_0 E$

Or $k^2 = \omega^2 \mu_0 \epsilon_0$

Or $v^2 = 1/\mu_0 \epsilon_0$ as $k = w/v$

Or $v = 1/\sqrt{\mu_0 \epsilon_0}$

Putting the value of $\mu_0 = 4\pi \times 10^7$ H/m and $\epsilon_0 = 8.85 \times 10^{-12}$ F/m

We get **$v = c = 3.00 \times 10^8$ m/s^2** …………9

So we can write the wave equation for electric and magnetic field as

$\nabla^2 E = 1/v^2 (\partial^2 E/\partial t^2)$…………………10

or $\nabla^2 B = 1/v^2 (\partial^2 B/\partial t^2)$………………11

Comparing equations 9 and 11 with standard wave equation propagating in x direction in free (vacuum) space $\{\partial^2 y/\partial x^2 = 1/v^2 (\partial^2 y/\partial t^2)\}$we get electromagnetic wave consist of electric and magnetic field both are perpendicular to propagation vector k with a speed $v = c = 3.00 \times 10^8$ m/s (speed of light in vacuum) in this condition we can write the wave equation and solution as

$\partial^2 E_y/\partial x^2 = \mu_0 \epsilon_0 (\partial^2 E_y/\partial t^2)$; $E(r, t) = E_0 \cdot e^{i(\omega t - k x)}$; $E = E_0 \sin(\omega t - k x)$

$\partial^2 B_z/\partial x^2 = \mu_0 \epsilon_0 (\partial^2 B_z/\partial t^2)$; $B(r, t) = B_0 \cdot e^{i(\omega t - k x)}$; $B = B_0 \sin(\omega t - k x)$

Basically, change of magnetic field produces electric field and change in magnetic field produces magnetic field and so wave is propagated along a particular axes say x direction and electric and magnetic field varies in y or z direction. Equation 9 reveals that this velocity is same as the velocity of light in free space. So we can conclude that light is simply a form of electromagnetic radiation. X-rays, ultraviolet rays, infrared, radio and microwave radiation are all electromagnetic radiation, differing only in the magnitude of their wavelength. Later prediction of electromagnetic waves proposed by Maxwell was verified experimentally by Heinritch Hertz 1n 1967. In the famous Hertz experiment wave was generated by oscillatory spark discharge and this wave follows the familiar properties of light such as reflection refraction, interference and diffraction.