Orthogonality of Hermite polynomials

We have the Hermite differential equation as

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0 \quad \text{— (1)}$$

As $H_n(x)$ is the solution of Hermite equation, we can write the eqn (1) as

$$\frac{d^2H_n(x)}{dx^2} - 2x\frac{d}{dx}H_n(x) + 2nH_n(x) = 0 \quad \text{— (2)}$$

Multiplying both sides by e^{-x^2} we get

$$\frac{d}{dx}\left[e^{-x^2}\frac{dH_n}{dx}\right] = -2n\left[e^{-x^2}H_n(x)\right] \quad \text{— (3)}$$

Also, $$\frac{d}{dx}\left[e^{-x^2}\frac{dH_m}{dx}\right] = -2m\left[e^{-x^2}H_m(x)\right], \text{ for } m \neq n \quad \text{— (4)}$$

Now,

(3) $\times H_m$ and (4) $\times H_n$ and

(3) $\times H_m$ − (4) $\times H_n$ we get

$$H_m(x)\frac{d}{dx}\left(e^{-x^2}\frac{dH_n}{dx}\right) - H_n(x)\frac{d}{dx}\left(e^{-x^2}\frac{dH_m}{dx}\right)$$
$$= 2(m-n)\,e^{-x^2}H_m(x)H_n(x) \quad \text{— (5)}$$

Now, integrating eqn (5) w.r.t x from $-\infty$ to $+\infty$ we get

$$\int_{-\infty}^{+\infty}\left\{H_m(x)\frac{d}{dx}\left(e^{-x^2}\frac{dH_n}{dx}\right) - H_n(x)\frac{d}{dx}\left(e^{-x^2}\frac{dHm}{dx}\right)\right\}dx$$
$$= 2(m-n)\int_{-\infty}^{+\infty} e^{-x^2}H_m(x)\,H_n(x)\,dx \quad \text{— (6)}$$

The LHS of the equation (6) is

$$\int_{-\infty}^{+\infty} \frac{d}{dx}\left\{H_m(x)\left(e^{-x^2}\frac{dH_n}{dx}\right)\right\}dx - \int_{-\infty}^{+\infty}\frac{d}{dx}\left\{H_n(x)\left(e^{-x^2}\frac{dH_m}{dx}\right)\right\}dx = 0$$

$$\therefore \int_{-\infty}^{+\infty} e^{-x^2} H_m(x) H_n(x) dx = 0, \quad m \neq n$$

So, Hermite polynomials are orthogonal to e^{-x^2}

— x —

Q: Prove that $\int_{-\infty}^{+\infty} e^{-x^2} H_n^2(x)\,dx = 2^n\sqrt{\pi}.n!$

Proof: we have the generating function

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \quad \text{—— (1)}$$

Differentiating (1) partially w.r.t x we get

$$2(x-t)e^{2tx-t^2} = \sum_{n=1}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!}$$

$$\text{or, } 2x\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} - 2\sum_{n=0}^{\infty} H_n(x)\frac{t^{n+1}}{n!} = \sum_{n=1}^{\infty} H_n(x)\frac{t^{n+1}}{(n-1)!}$$

Equating the co-efficient of t^n from both side we get

$$2x\frac{H_n(x)}{n!} - 2\frac{H_{n-1}(x)}{(n-1)!} = \frac{H_{n+1}(x)}{n!}$$

$$\text{or, } H_{n+1} - 2xH_n + 2nH_{n-1} = 0 \quad \text{—— (2)}$$

Now, replacing n by $(n-1)$ we get

$$H_n - 2xH_{n-1} + 2(n-1)H_{n-2} = 0 \quad \text{—— (3)}$$

$$\text{or } H_n^2 - 2xH_nH_{n-1} + 2(n-1)H_nH_{n-2} = 0 \quad \text{—— (4)}$$

[obtained (3) × H_n]

Now, ② $\times H_{n-1}$ we get

$$H_{n-1}H_{n+1} - 2x H_n H_{n-1} + 2n H_{n-1}^2 = 0 \quad \text{— ⑤}$$

Again ④ − ⑤, we get

$$H_n^2 - H_{n-1}H_{n+1} + 2(n-1) H_n H_{n-2} - 2n H_{n-1}^2 = 0 \quad \text{— ⑥}$$

Now, ⑥ $\times e^{-x^2}$ and then integrating w.r.t x from $-\infty$ to $+\infty$, we get

$$\int_{-\infty}^{+\infty} e^{-x^2} H_n^2 dx - \int_{-\infty}^{+\infty} e^{-x^2} H_{n+1} H_{n-1} dx + 2(n-1)\int_{-\infty}^{+\infty} e^{-x^2} H_n H_{n-2} dx$$
$$- 2n \int_{-\infty}^{+\infty} e^{-x^2} H_{n-1}^2 dx = 0 \quad \text{— ⑦}$$

Using orthogonal property we get

$$\int_{-\infty}^{+\infty} e^{-x^2} H_n^2(x)\, dx = 2n \int_{-\infty}^{+\infty} e^{-x^2} H_{n-1}^2(x)\, dx$$
$$= 2^2 n(n-1) \int_{-\infty}^{+\infty} e^{-x^2} H_{n-2}^2(x)\, dx$$
$$= 2^n . n! \int_{-\infty}^{+\infty} e^{-x^2} H_0^2(x)\, dx$$
$$= 2^n . n! \int_{-\infty}^{+\infty} e^{-x^2} dx \quad \left[\because H_0(x) = 1\right]$$
$$= 2^n . n! . 2 \int_0^{\infty} e^{-x^2} dx$$
$$= 2^n . n! . 2 . \frac{\sqrt{\pi}}{2}$$
$$= 2^n . n! \sqrt{\pi} \quad \text{— ⑧}$$

Combining eqn $\int_{-\infty}^{+\infty} e^{-x^2} H_m(x) H_n(x)\, dx = 0$ and eqn ⑧ we get

$$\int_{-\infty}^{+\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n . n! \sqrt{\pi}\, \delta_{m,n}$$

where $\delta_{m,n} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$

— x —

(8)

Q: Prove the recurrence relation

① $nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2}$

(2) $xP_n' - P_{n-1}' = nP_n$, ③ $P_n' - xP_{n-1}' = nP_{n-1}$

④ $2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$

Solⁿ ① we have $(1-2xy+y^2)^{-1/2} = \sum y^n P_n(x)$

Differentiating partially, w.r.t y we get

$$-\frac{1}{2}(1-2xy+y^2)^{-3/2} \cdot (-2x+2y) = \sum n y^{n-1} P_n(x)$$

Multiplying both side by $(1-2xy+y^2)$,

$$(1-2xy+y^2)^{-1/2}(x-y) = (1-2xy+y^2)\sum n y^{n-1} P_n(x)$$

$$\Rightarrow (x-y)\sum y^n P_n(x) = (1-2xy+y^2)\sum x y^n P_n(x)$$

Equaling from both side, the ~~equ~~ co-efficient of y^{n-1} we get

$$xP_{n-1} - P_{n-2} = nP_n - 2x(n-1)P_{n-1} + (n-2)P_{n-2}$$

$$\Rightarrow \boxed{nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2}}$$

(2) We have $(1-2xy+y^2)^{-1/2} = \sum y^n P_n(x)$ — (1) (5)

Differentiating (1) w.r.to y we get

$$-\frac{1}{2}(1-2xy+y^2)^{-3/2}(-2x+2y) = \sum n y^{n-1} P_n(x)$$

$$\Rightarrow (x-y)(1-2xy+y^2)^{-3/2} = \sum n y^{n-1} P_n(x) \quad \text{— (2)}$$

Again Differentiating (1) w.r.to x we get

$$-\frac{1}{2}(1-2xy+y^2)^{-3/2}(-2y) = \sum y^n P_n'(x)$$

$$\Rightarrow y(1-2xy+y^2)^{-3/2} = \sum y^n P_n'(x) \quad \text{— (3)}$$

Now, (2) ÷ (3) we get

$$\frac{x-y}{y} = \frac{\sum n y^{n-1} P_n(x)}{\sum y^n P_n'(x)}$$

$$\Rightarrow (x-y)\sum y^n P_n'(x) = \sum n y^n P_n(x)$$

$$\Rightarrow x\sum y^n P_n'(x) - \sum y^{n+1} P_n'(x) = \sum n y^n P_n(x)$$

$$\Rightarrow x\sum y^n P_n'(x) - \sum y^n P_{n-1}'(x) = \sum n y^n P_n(x)$$

Equating the co-efficient of y^n from both side we get $\boxed{x P_n'(x) - P_{n-1}'(x) = n P_n(x)}$

(3) is obtained by differentiating (w.r.to x) recurrence relation (1) given and using recurrence relation (2) given.

(4) We have

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$$

Differentiating partially w.r.to 't' we get

$$2(x-t)\,e^{2tx-t^2} = \sum_{n=1}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!}$$

$$\Rightarrow 2x\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} - 2\sum_{n=0}^{\infty} H_n(x)\frac{t^{n+1}}{n!} = \sum_{n=1}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!}$$

Equating the co-efficient of t^n from both sides we get

$$2x\,\frac{H_n(x)}{n!} - 2\,\frac{H_{n-1}(x)}{(n-1)!} = \frac{H_{n+1}(x)}{n!}$$

$$\Rightarrow \boxed{2x\,H_n(x) = 2n\,H_{n-1}(x) + H_{n+1}(x)}$$