Generating function of $H_n(x)$:

The generating function of Hermite polynomial $H_n(x)$, for real values of x and integral n is given by

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \quad —(1)$$

Proof: We have

$$e^{2tx-t^2} = e^{2tx} \cdot e^{-t^2}$$
$$= \sum_{r=0}^{\infty} \frac{(2tx)^r}{r!} \sum_{s=0}^{\infty} \frac{(-t^2)^s}{s!}$$
$$= \sum_{r,s=0}^{\infty} (-1)^s \frac{(2x)^r}{r!\,s!} t^{r+2s}$$

∴ The co-efficient of t^n for fixed s is

$$(-1)^s \cdot \frac{(2x)^{n-2s}}{(n-2s)!\,s!} \quad \left[\text{putting } r+2s=n\right.$$

The total co-efficient of t^n can be obtained by summing over all the allowed values of s,

since $r = n-2s \geq 0$, or $s \leq \frac{n}{2}$

so, if n is even, $s \to 0$ to $\frac{n}{2}$

and if n is odd, $s \to 0$ to $\frac{1}{2}(n-1)$

So, the required co-efficient of t^n is

$$\sum_{s=0}^{n/2} (-1)^s \frac{(2x)^{n-2s}}{(n-2s)!\,s!} = \frac{H_n(x)}{n!}$$ | using Hermite Polynomial for n even

$$\therefore\ e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

$$\text{or, } e^{x^2-(t-x)^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

— x —

<u>Rodrigue's Formule for $H_n(x)$:-</u>

We have found from generating function for $H_n(x)$

$$e^{x^2-(t-x)^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \quad \text{— (1)}$$

Differentiating both side of eqn (1) w.r.to t n times and putting $t=0$ we get

$$\frac{H_n(x)}{n!}\, n! = \left[\frac{\partial^n}{\partial t^n} \cdot e^{-(t-x)^2}\right]_{t=0} e^{x^2}.$$

Putting $u = t-x$, when $x=-u$ as $t=0$, so that $\frac{\partial}{\partial t} = \frac{\partial}{\partial u}$

we get

$$\left[\frac{\partial^n}{\partial x^n} e^{-(t-x)^2}\right]_{t=0} = \frac{\partial^n}{\partial u^n}\left(e^{-u^2}\right) = (-1)^n \frac{\partial^n}{\partial x^n}\left(e^{-x^2}\right)$$

$$= (-1)^n \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

$$\therefore\ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \quad \text{— (2)}$$

Equation ② is the differential form of Hermite Polynomial and is called Rodrigue's formula.

Putting $n = 0, 1, 2 \cdots$ in equation ② we get

$$H_0(x) = e^{x^2} \cdot e^{-x^2} = 1$$

$$H_1(x) = (-1) e^{x^2} \frac{d}{dx}(e^{-x^2}) = 2x$$

$$H_2(x) = (-1) e^{x^2} \frac{d^2}{dx^2}(e^{-x^2})$$

$$= e^{x^2} \frac{d}{dx}(-2x e^{-x^2})$$

$$= 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12 \text{ and so on.}$$

— x —

<u>Recurrence formula for $H_n(x)$:</u>

we have the generating function for $H_n(x)$

$$e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \quad \text{— ①}$$

Differentiating w.r.to x we get

$$2t \cdot e^{2tx - t^2} = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!}$$

$$\Rightarrow 2t \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!} \quad \left[\frac{d}{dx} H_n(x) = H_n'\right]$$

Equating the co-efficient of $\frac{t^n}{n!}$ from both side we get

$$2 \cdot \frac{n!}{(n-1)!} H_{n-1}(x) = H_n'(x)$$

$$n = \frac{n!}{(n-1)!}$$

$$\text{or } 2n H_{n-1}(x) = H_n'(x)$$

— x —