Hermite's Equation :-

Hermite differential equation is.

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0 \quad \text{— (1)}$$

where n is a non-negative integer.

The power series solution is

$$y = \sum_{r=0}^{\infty} a_r x^{m+r} \quad \text{— (2)}$$

$$\therefore \frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (m+r) x^{m+r-1}$$

and $$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2}$$

substituting these in eqn (1) we get

$$\sum_{r=0}^{\infty} \left[(m+r)(m+r-1)x^{m+r-2} - 2(m+r)x^{m+r} + 2nx^{m+r}\right] a_r = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} \left[(m+r)(m+r-1)x^{m+r-2} - 2(m+r-n)x^{m+r}\right] a_r = 0 \quad \text{— (3)}$$

Equating the co-efficient of the lowest power of x i.e x^{m-2} to zero [taking $r=0$] we get

$$a_0[m(m-1)] = 0 \quad \text{— (4)}$$

$\because a_0 \neq 0$ so, $m(m-1) = 0$

$\therefore m = 0$ or $m = 1$

Again equating the co-efficient of x^{m-1} to zero (taking $r=1$) we get

$$a_1[m(m+1)] = 0 \quad \text{— (5)}$$

Now,
equating the co-efficient of the general term x^{m+r} to zero we get

$$a_{r+2}(m+r+2)(m+r+1) - 2a_r(m+r-n) = 0$$

$$\Rightarrow a_{r+2} = \frac{2(m+r-n)}{(m+r+2)(m+r+1)} a_r$$

$$= \frac{2(m+r)-2n}{(m+r+2)(m+r+1)} a_r \rightarrow (6)$$

This is the recurrence relation of the co-efficients.

Case: 1 When $m=0$

From equation (6) we get

$$a_{r+2} = \frac{2r-2n}{(r+2)(r+1)} a_r$$

$$\therefore a_2 = -\frac{2n}{2!}, \quad a_4 = \frac{4-2n}{4 \cdot 3} a_2$$

$$= -\frac{(4-2n) \cdot 2n}{4 \cdot 3 \cdot 2!} a_0$$

$$= \frac{+2(-2+n)2n \cdot a_0}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \frac{2^2(-2+n)n}{4 \cdot 3 \cdot 2 \cdot 1} a_0 = \frac{2^2(-2+n)n}{4!} a_0$$

$$\therefore a_{2m} = \frac{(-2)^K n(n-2) \cdots\cdots (n-2K+2)}{(2K)!} a_0$$

Similarly,

$$a_3 = \frac{2-2n}{3 \cdot 2} a_1 = -\frac{2(n-1)}{3!} a_1$$

$$a_5 = \frac{6-2n}{5 \cdot 4} a_3 = -\frac{2(n-1)(6-2n)}{5 \cdot 4 \cdot 3!}$$

$$= (-2)^2 \frac{(n-1)(n-3)}{5!} a_1 \text{ etc.}$$

The general term is

$$a_{2K+1} = \frac{(-2)^K (n-1)(n-3) \cdots (n-2K+1)}{(2K+1)!}$$

$$\therefore y = \sum_{r=0}^{\infty} a_r x^{m+r}$$

$$= \sum_{r=0}^{\infty} a_r x^r \quad [\because m=0$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$\Rightarrow = a_0 \left[1 - \frac{2n}{2!} x^2 + \frac{2^2 n(n-2)}{4!} x^4 + \cdots + \cancel{(-2)^K n (n-2)} \right.$$

$$\left. + (-2)^K \frac{n(n-2) \cdots (n-2K+2)}{2K!} x^{2K} \right]$$

$$+ a_1 \left[x - \frac{2(n-1)}{3!} x^3 + 2^2 \frac{(n-1)(n-3)}{5!} x^5 + \cdots \right] \quad \because a_1 \neq 0 \rightarrow (7)$$

and $$y = a_0 \left[1 - \frac{2n}{2!} x^2 + \frac{2^2 n(n-2)}{4!} x^4 + \cdots \right] \text{ for } a_1 = 0$$

$$= u \text{ (considered)} \text{ —— } (8)$$

Case II: when $n=1$

From equation (6) we get

$$a_{r+2} = \frac{2(r+1)-2n}{(r+3)(r+2)} a_r$$

Putting $r = 1, 3, 5 \ldots$ we get

$$a_3 = a_5 = \cdots = 0 \quad (\because a_1 = 0)$$

also putting $r = 0, 2, 4 \ldots$ etc we get

$$a_2 = \frac{2-2n}{3.2} a_0 = -\frac{2(n-1)}{3!} a_0$$

$$a_4 = \frac{6-2n}{5.4} a_2 = \frac{2^r (n-1)(n-3)}{5!} a_0$$

and so on.

The general form is

$$a_{2K} = (-1)^K \frac{(n-1)(n-3)\cdots(n-2K+1)}{(2K+1)!}$$

$$\therefore y = \sum_{r=0}^{\alpha} a_r x^{r+1}$$

$$= a_0 x + a_2 x^3 + a_4 x^5 + \cdots + a_{2K} x^{2K+1}$$

or $y = a_0 \left[x - \frac{2(n-1)}{3!} x^3 + \frac{2^r (n-1)(n-3)}{5!} x^5 + \cdots \right] \rightarrow (9)$

$= v$ (considered)

$\therefore$ The general solution is

$$y = Au + Bv$$

— x —

Hermite Polynomial $H_n(x)$

Taking $a_0 = (-1)^{n/2} \dfrac{n!}{\left(\frac{n}{2}\right)!}$ and n is even,
the co-efficient of x^n in equation (*)

$$y = a_0 \left[1 - \frac{2n}{2!} x^2 + \frac{2^2 n(n-2)}{4!} x^4 + \cdots \right]$$ [Sol^n of H. equation when $n \neq 0$]

will be

$$(-1)^{n/2} \frac{n!}{(n/2)!} \cdot \frac{(-2)^{n/2} n(n-2)\cdots(n-n+2)}{n!}$$

$$= 2^n \frac{\frac{n}{2}\cdot\left(\frac{n}{2}-1\right)\cdots 1}{(n/2)!}$$

$$= 2^n$$

$^{||}$ly, the co-efficient of x^{n-2} is given by

$$(-1)^{n/2} \frac{n!}{(n/2)!} (-2)^{(n-2)/2} n(n-2)\cdots(n-n+2+2)$$

$$= -\frac{n(n-1)}{1!} 2^{n-2}, \text{ etc}$$

$$\therefore y_n = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} + \cdots + (-1)^{n/2} \frac{n!}{(n/2)!}$$

$$= H_n(x)$$

Where $H_n(x)$ is the polynomial y_n and is called the Hermite polynomial of degree n.

⑥

$$\therefore\ H_n(x) = \begin{cases} \displaystyle\sum_{r=0}^{\alpha} (-1)^r \frac{n!}{r!\,(n-2r)!} (2x)^{n-2r}, & n \to \text{even} \\[2ex] \displaystyle\sum_{r=0}^{\frac{n}{2}-1} (-1)^r \cdot \frac{n!}{r!\,(n-2r)!} (2x)^{n-2r}, & n \to \text{odd} \end{cases}$$

— x —