Q. 5.8. What is photon gas? What is the difference between photon gas and ideal gas? Starting from Bose-Einstein energy distribution law derive Planck's law of black body radiation.

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G.N.D.U. 2008, 2006, 2002, 2001, 2000; *Pbi.U.* 2005, 2002, 2001;
Meerut.U. 2006, 2005, 2001, 1999, *M.D.U.* 2001; *Poorvanchal.U.* 2001;
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Ans. Photon gas. Planck's law of distribution of energy with wavelength in the case of black body radiations can be derived on the basis of Bose-Einstein statistics.

If we have a hollow enclosure the walls of which are maintained at a constant temperature *T* and a small hole is made in the walls of the enclosure, the radiation coming out will be those which a perfectly black body will emit at a temperature *T*. The atoms of the walls of the enclosure emit electromagnetic radiations and at the same time these radiations are absorbed by the atoms in the walls. Thus the atoms in the walls are continuously emitting and absorbing radiation. Finally, the radiations reach an equilibrium with the atoms of the walls. At this stage the amount of energy emitted by the atoms per unit time is equal to the amount of energy absorbed by the atoms per unit time. This equilibrium radiation in the enclosure is black body radiation at temperature T.

According to classical theory the frequency of these radiation lies continuously between zero and infinity. But it has been observed that these radiations carry energy in discrete units or bundles or *quanta*. According to *quantum theory*, radiation of frequency ν has a quantum of enrgy $h\nu$ where h is Planck's constant and momentum $\frac{h\nu}{c}$ where c is the velocity of light. These quanta are known as photons and can be treated as particles. The radiations inside the hollow enclosure consists of a very large number of photons of different energies as these have different wavelengths (or frequencies) and can be supposed to form a *photon gas*. The distribution of energy among these photons inside the enclosure obeys the law of statistics. Since the photons have *integral spin* angular momentum in units of $\frac{h}{2\pi}$, they are bosons and obey Bose-Einstein statistics.

Difference between photon gas and ideal gas. An ideal gas differs from a photon gas in many respects.

(*i*) The rest mass of the photon is zero. A gas molecule has a finite rest mass.

(*ii*) The molecules of a perfect gas are distinguishable. The photons in a photon gas are indistinguishable from each other.

(*iii*) Photon gas obeys Bose-Einstein statistics whereas an ideal gas obeys Maxwell Boltzmann statistics.

(*iv*) Ideal gas molecules collide with each other and also with the walls of the containing vessel but no molecules is either absorbed or emitted by the walls. *Thus the total number of molecules in the gas remains constant.*

In case of photon gas, a photon can be absorbed or emitted by the walls of the containing vessel. *Thus total number of photons does not remain constant.*

(*v*) The total energy of an ideal gas as well as of a photon gas remains constant if the temperature remains constant.

Derivation of Planck's law from B.E. energy distribution law. For a system obeying Bose-Einstein statistics the condition for maximum thermodynamic probability is

$$d\,(\ln W) = \sum_{i=1}^{k}\left[\ln \frac{n_i + g_i}{n_i}\right] dn_i = 0 \qquad \text{[Q. 5.2 Eq. (iii)]} \ldots(i)$$

where W is the thermodynamic probability for the ith compartment to have n_i particles distributed in its g_i cells.

The case of photon gas differs from the ideal gas in one respect. Although the total energy of photons inside the hollow enclosure at a particular temperature T remains constant, the total number of photons within the enclosure may not remain constant. It is possible that a photon may be completely absorbed on striking the walls of the enclosure or the hot wall may emit a new photon or that a photon of energy $2h\nu$ may be absorbed and two photons each of energy $h\nu$ may be emitted *i.e.*, the photons may be *created* or *destroyed*. In other words for a *photon gas system* $\sum_{i=1}^{k} dn_i \neq 0$

Therefore, the system has to satisfy only one condition namely, the total energy of the system remains a constant or U = a constant $\therefore dU = \sum_{i=1}^{k} u_i\, dn_i = 0$

Multiplying the above relation with β and equating it to relation (*i*), since both are equal to zero, we have

$$\sum_{i=1}^{k}\left[\ln\frac{n_i+g_i}{n_i}\right]dn_i=\beta\sum_{i=1}^{k}u_i\,dn_i \quad \text{or} \quad \sum_{i=1}^{k}\left[\ln\frac{n_i+g_i}{n_i}-\beta u_i\right]dn_i=0$$

The above relation is satisfied if the values of the quantity within brackets are separately equal to zero for each value of i

$$\therefore \quad \ln\frac{n_i+g_i}{n_i}=\beta u_i=\frac{u_i}{kT} \qquad \left[\because \beta=\frac{1}{kT}\right]$$

or
$$1+\frac{g_i}{n_i}=e^{\frac{u_i}{kT}} \qquad \therefore\ n_i=\frac{g_i}{e^{\frac{u_i}{kT}}-1} \qquad ...(ii)$$

To find the number of photons in the frequency interval ν and (ν + dν) consider a unit interval of frequency of radiation lying between (ν – 1/2) and $\left(\nu+\frac{1}{2}\right)$ and let n(ν) be the number of photons in this frequency interval, then substituting $u_i=u=h\nu$ in (ii), we have $n(\nu)=\dfrac{g(\nu)}{e^{h\nu/kT}-1}$

The number of photons in the frequency interval ν and (ν + dν) is, then given by

$$n(\nu)\,d\nu=\frac{g(\nu)d\nu}{e^{h\nu/kT}-1} \qquad ...(iii)$$

In the above relation $g(\nu)\,d\nu$ gives the number of cells in the phase space volume lying in the energy interval (compartment) between $h\nu$ and $h(\nu+d\nu)$

The corresponding values of momentum are

$$p=\frac{h\nu}{c} \text{ and } p+dp=\frac{h(\nu+d\nu)}{c} \qquad \therefore\ dp=\frac{h}{c}\,d\nu$$

The number of cells in the phase space in the momentum interval between p and $(p+dp)$ is given by $g(p)\,dp=\dfrac{4\pi Vp^2\,dp}{h^3}$ in the case of ordinary gas molecules obeying M.B. statistics where V is the volume occupied by the gas molecules in ordinary position space. But in the case of photons obeying B.E. statistics due to the property of *polarisation* of photons on account of their wave character the number will be *twice* as much. It is because, the photons inside the constant temperature enclosure are of *two* types : (i) Those having *left*-handed polarisation and (ii) Those having *right*-handed polarisation. The photons of the first type are *indistinguishable* among themselves but can be distinguished from those of the second type on account of different polarisation. Similarly the photons of the second type are indistinguishable among themselves but can be distinguished from those of the first type on account of different polarisation. In other words, the whole system can be considered to be made up of two sub-systems, one having photons with left-handed polarisation and the other having an equal number of photons with right-handed polarisation. The B.E. statistics will be applied separately to these two sub-systems. The number of cells in the phase space for one kind of photons is given by $g(p)\,dp=\dfrac{4\pi Vp^2}{h^3}\,dp$

$\therefore$ The number of cells for both kind of photons, is given by $g(p)\,dp=\dfrac{8\pi Vp^2}{h^3}\,dp$

Substituting the value of $p=\dfrac{h\nu}{c}$ and $dp=\dfrac{h}{c}\,d\nu$, we have

$$g(\nu)\, d\nu = \frac{8\pi V}{h^3} \frac{h^2 \nu^2}{c^2} \cdot \frac{h}{c}\, d\nu = \frac{8\pi V}{c^3} \nu^2\, d\nu$$

Substituting in relation (*iii*), we have $n(\nu)\, d\nu = \frac{8\pi V}{c^3} \nu^2\, d\nu \frac{1}{e^{h\nu/kT} - 1}$ ...(*iv*)

Since each photon has energy $h\nu$, the *energy density* $E(\nu)\, d\nu$ defined as the amount of energy per unit volume lying between the frequencies ν and $(\nu + d\nu)$ is given by $\frac{h\nu\, n(\nu)\, d\nu}{V}$

$$\therefore \quad E(\nu)\, d\nu = \frac{h\nu\, n(\nu)\, d\nu}{V} = \frac{8\pi h\nu^3\, d\nu}{c^3} \frac{1}{e^{h\nu/kT} - 1} \qquad ...(v)$$

Equation (ν) *is known as Planck's law of black body radiation in terms of frequency of radiation.* This gives the energy density of radiation having frequency between ν and $\nu + d\nu$

Planck's law in terms of wavelength. To express the law in terms of wavelength (λ) substituting $\nu = \frac{c}{\lambda}$ and $d\nu = -\frac{c}{\lambda^2} d\lambda$, in Eq. (*v*), we get

$$E_\lambda\, d\lambda = \frac{8\pi h}{c^3} \cdot \frac{c^3}{\lambda^3} \cdot \frac{c\,d\lambda}{\lambda^2} \frac{1}{e^{hc/\lambda kT} - 1} = \frac{8\pi\, hc}{\lambda^5} \frac{d\lambda}{e^{hc/\lambda kT} - 1}$$

or

$$|E_\lambda| = \frac{8\pi hc\, \lambda^{-5}}{e^{hc/\lambda kT} - 1} \qquad ...(vi)$$

Equation (*vi*) *is known as Planck's law for black body radiation in terms of wavelength.*

Planck's law accurately fits in the experimental results regarding the distribution of energy with wavelength in the spectrum due to radiations from a black body.