Q. What is Fermi-Energy? Calculate the value of Fermi-energy of a metal. Does the Fermi-energy depend upon size or volume of a conductor.

Ans: The energy values up to which all the energy states are full at 0°K and above which all the energy states are empty is known as Fermi-energy.

Value of Fermi-energy :-

At the temperature T=0K, the number of electrons is equal to the total number of energy states occupied by the electrons from zero to u_f, since each energy state can have one electron.

i.e

$$n = \int_0^{u_f} g(u)\,du = \frac{8\sqrt{2}\,\pi V}{h^3} m^{3/2} \int_0^{u_f} u^{1/2}\,du$$

$$= \frac{8\sqrt{2}\,\pi V m^{3/2}}{h^3} \left[\frac{2}{3} u^{3/2}\right]_0^{u_f}$$

$$= 16\sqrt{2}\,\pi V \frac{m^{3/2}}{3h^3} u_f^{3/2}$$

$$\therefore \boxed{u_f = \frac{h^2}{2m}\left(\frac{3n}{8\pi V}\right)^{2/3}}$$

It is clear from the above expression of fermi energy that it is independent of the size and volume of of the conductor as it only depends upon $\left(\frac{n}{V}\right)$, the number of electron per unit volume.

—x—

√ **Q. 5.2. What is Bose-Einstein statistics? What difficulties were met with M.B. statistics and how they were explained? Write basic assumptions of Bose-Einstein quantum statistics and derive the expression $n_i = \frac{g_i}{e^{\alpha} e^{u_i/kT} - 1}$ for the most probable distribution of a system of particles obeying B.E. statistics**

(*Pbi. U.* 2007; *K.U.*, 2002, 2001, 2000; *H.P.U.* 2001, 1994; *Madurai. U.*, 2003; *M.D.U.*, 2005, 2000; *P.U.*, 1999, 1994; *G.N.D.U.*, 2004, 1996, 1995)

Ans. Bose-Einstein statistics. Bose-Einstein statistics is obeyed by those particles which are identical, indistinguishable and have integral spin. These particles are known as *Bosons*.

Bose used Planck's hypothesis according to which radiation in a temperature enclosure are composed of light quanta or photons of energy $h\nu$. These photons in the enclosure are indistinguishable.

Difficulties with M.B. statistics. See need for quantum statistics **Q. 5.1.**

Assumptions. In Bose-Einstein statistics following assumptions are made:

(*i*) The particles of the system are *identical* and *indistinguishable.*

(*ii*) *Any number of particles can occupy a single cell in the phase space.*

(*iii*) The size of the cell cannot be less than h^3 where h is Planck's constant having a value $6{\cdot}63 \times 10^{-34}$ Joule sec.

(*iv*) The number of phase space cells is comparable with the number of particles *i.e.*, occupation index $\frac{n_i}{g_i} = 1$.

(*v*) Bose-Einstein statistics is applicable to particles with *integral spin* angular momentum in units of $\frac{h}{2\pi}$. All particles which obey B.E statistics one known as *Bosons*.

Bose-Einstein distribution law. Consider a system consisting of n independent identical particles. The particles have definite energies or *momenta* as well as occupy definite *positions* and hence can be represented by *phase* points in the phase space. In order to determine the energy or momentum distribution of these particles in the *most probable* or *equilibrium state* we divide the available volume in the phase space into a large number (say k) compartments (*quantum group* or *energy levels*) each compartment representing a small interval of energy (or momentum). Further we divide each compartment into elementary cells of size h^3 where h is Planck's constant. We further suppose that the size of the compartment is very large as compared to the size of the cell so that each compartment contains a *very large number* of elementary cells.

Let there be $n_1, n_2, \ldots\ldots n_i \ldots\ldots n_k$ particles having mean energy values $u_1, u_2, \ldots\ldots\ldots u_i \ldots\ldots\ldots u_k$ respectively in compartments numbered as 1, 2, ... i.........k containing $g_1, g_2 \ldots\ldots g_i \ldots\ldots g_k$ cells respectively in them.

The total number of particles in the system $n = n_1 + n_2 + \ldots.. n_i + \ldots\ldots n_k$

Now consider the ith compartment. It has n_i *indistinguishable* particles distributed among its g_i cells. Let us calculate the number of ways in which these n_i *identical* particles can be distributed among g_i cells. To do this, we *first* choose the cell with which to begin the sequence. This choice can be made in g_i ways since there are g_i cells. Once this has been done, the remaining $(g_i - 1)$ cells and n_i particles *i.e.*, total $(n_i + g_i - 1)$ *elements* can be arranged in order. The number of ways of doing this $= (n_i + g_i - 1)!$

Thus the total number of ways for realising the distribution will be $g_i (n_i + g_i - 1)!$

As the particles are *indistinguishable*, the re-arrangement of particles among themselves will not give rise to any new distribution. The number of such *meaningless* permutations is n_i! Hence the above result should be divided by n_i!

Secondly, the distribution which can be derived by mere permutation of g_i cells among themselves $= g_i$! These permutations also do not produce different states and are hence *meaningless*. The above result should, therefore, be further divided by g_i!

Thus the number of different arrangements or *meaningful microstates* for the ith compartment having n_i *indistinguishable* particles to be distributed among its g_i cells, is given by

$$W_i = \frac{g_i\,(n_i + g_i - 1)!}{n_i!\,(g_i)!}$$

Now since $g_i! = g_i\,(g_i - 1)!$ we can write the above relation as

$$W_i = \frac{(n_i + g_i - 1)!}{n_i!\,(g_i - 1)!} \qquad \ldots(i)$$

Equation (i) gives the number of different arrangements of distributing n_i distinguishable particles anong g_i cells in the ith compartment.

Similar expressions will be obtained for other compartments. Therefore, the total number of different arrangements for all the n particles of the system gives thermodynamic probability

$$W(n_1, n_2 \ldots n_i \ldots n_k) = \frac{(n_1 + g_1 - 1)!}{n_1!\,(g_1 - 1)!} \times \frac{(n_2 + g_2 - 1)!}{n_2!\,(g_2 - 1)!} \ldots\times \frac{(n_i + g_i - 1)!}{n_i!\,(g_i - 1)!} \ldots\times \frac{(n_k + g_k - 1)!}{n_k!\,(g_k - 1)!}$$

$$= \prod_{i=1}^{k} \frac{(n_i + g_i - 1)!}{n_i!\,(g_i - 1)!} \qquad \ldots(ii)$$

where Π denotes multiplication of terms stated above for various values of i.

Most probable macrostate. The most *probable* macrostate corresponds to the state of maximum thermodynamic probability.

In Eq. (*ii*) n_i and g_i are both very large numbers so that neglecting 1 as compared to $(n_i + g_i)$ and g_i, we get

$$W(n_1, n_2 \ldots n_i \ldots n_k) = \prod_{i=1}^{k} \frac{(n_i + g_i)!}{n_i!\,g_i!}$$

Taking natural logarithms on both sides, (and representing $W(n_1, n_2 \ldots n_i \ldots n_k)$ simply as W, we have

$$\ln W = \sum_{i=1}^{k} [\ln (n_i + g_i)! - \ln n_i! - \ln g_i!]$$

Applying Stirling's approximation we get

$$\ln W = \sum_{i=1}^{k} [(n_i + g_i) \ln (n_i + g_i) - (n_i + g_i) - n_i \ln n_i + n_i - g_i \ln g_i + g_i]$$

To get the state of *maximum thermodynamic* probability, we differentiate the expression for $\ln W$ and equate it to zero.

$$\therefore \quad d(\ln W) = \sum_{i=1}^{k}\left[dn_i \ln (n_i + g_i) + (n_i + g_i)\frac{1}{(n_i + g_i)} dn_i - dn_i \ln n_i - n_i \cdot \frac{1}{n_i} dn_i \right]$$

$$= \sum_{i=1}^{k} dn_i \ln (n_i + g_i) - dn_i \ln n_i \qquad [\because g_i \text{ is a mere number } d(g_i) = 0]$$

or $$d(\ln W) = \sum_{i=1}^{k}\left[\ln \frac{(n_i + g_i)}{n_i}\right] dn_i = 0 \qquad ...(iii)$$

In addition our system must satisfy two auxillary conditions.

(*i*) *Conservation of total number of particles i.e.*, n = a constant.

$$\therefore \quad dn = \sum_{i=1}^{k} dn_i = 0 \qquad ...(iv)$$

(*ii*) *Conservation of total energy of the system i.e.*, U = a constant

$$\therefore \quad dU = \sum_{i=1}^{k} u_i \, dn_i = 0 \qquad ...(v)$$

Multiplying equation (*iv*) with α and equation (*v*) with β and adding, we get

$$\alpha \sum_{i=1}^{k} dn_i + \beta \sum_{i=1}^{k} u_i \, dn_i = 0 \qquad ...(vi)$$

Equating relation (*iii*) and (*vi*) we get

$$\sum_{i=1}^{k}\left[\ln \frac{(n_i + g_i)}{n_i}\right] dn_i = \alpha \sum_{i=1}^{k} dn_i + \beta \sum_{i=1}^{k} u_i \, dn_i$$

or $$\sum_{i=1}^{k}\left[\ln \frac{(n_i + g_i)}{n_i} - \alpha - \beta u_i\right] dn_i = 0$$

As the various dn_i's are independent of one another, the above sum is equal to zero if the expression within the brackets vanishes separately for each value of i

$$\therefore \quad \ln \frac{n_i + g_i}{n_i} = \alpha + \beta u_i = \alpha + \frac{u_i}{kT} \qquad \left[\because \beta = \frac{1}{kT}\right]$$

or $$\frac{n_i + g_i}{n_i} = e^{\alpha} e^{\frac{u_i}{kT}} \quad \text{or} \quad 1 + \frac{g_i}{n_i} = e^{\alpha} e^{\frac{u_i}{kT}}$$

$$\therefore \quad \frac{g_i}{n_i} = e^{\alpha} e^{\frac{u_i}{kT}} - 1 \quad \text{or} \quad n_i = \frac{g_i}{e^{\alpha} e^{\frac{u_i}{kT}} - 1} \qquad ...(vii)$$